

# SQUARE-FREE VALUES OF MULTIVARIATE POLYNOMIALS OVER FUNCTION FIELDS IN LINEAR SPARSE SETS

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**ABSTRACT.** Let  $f \in \mathbb{F}_q[t][x]$  be a square-free polynomial where  $\mathbb{F}_q$  is a field of  $q$  elements. We view  $f$  as a polynomial in the variable  $x$  with coefficients in the ring  $\mathbb{F}_q[t]$ . We study square-free values of  $f$  in sparse subsets of  $\mathbb{F}_q[t]$  which are given by a linear condition. The motivation for our study is an analogue problem of representing square-free integers by integer polynomials, where it is conjectured that setting aside some simple exceptional cases, a square-free polynomial  $f \in \mathbb{Z}[x]$  takes infinitely many square-free values. Let  $\kappa \in \mathbb{N}$  be co-prime to  $q$ , and let  $\gamma_1, \dots, \gamma_{\kappa-1}, \gamma_{\kappa+1}, \dots, \gamma_m \in \mathbb{F}_q$ . A consequence of the main result we show, is that if  $q$  is sufficiently large with respect to  $\deg_x f, \deg_t f$  and  $m$ , then there exist  $\gamma_0, \gamma_\kappa \in \mathbb{F}_q$  such that  $f(t, \sum_{i=0}^m \gamma_i t^i)$  is square-free. Moreover, as  $q \rightarrow \infty$ , the last is true for almost all  $\gamma_0, \gamma_\kappa \in \mathbb{F}_q$ . The main result shows that a similar result holds also for other cases. We then generalize the results to multivariate polynomials.

## 1. INTRODUCTION

Let  $f \in \mathbb{F}_q[t][x]$  where  $\mathbb{F}_q$  is a field of  $q$  elements and  $p = \text{Char}(\mathbb{F}_q)$ . We consider  $f$  as a univariate polynomial in  $x$  where its coefficients lay in the ring  $\mathbb{F}_q[t]$ . The result of substituting the variable  $x$  with an element in the base ring  $\mathbb{F}_q[t]$ , is a polynomial in  $\mathbb{F}_q[t]$ , i.e. for any  $u \in \mathbb{F}_q[t]$ ,  $f(t, u(t)) \in \mathbb{F}_q[t]$ . A polynomial is said to be square-free if it does not have a nonconstant square divisor. If there exists  $u \in \mathbb{F}_q[t]$  such that  $f(t, u(t))$  is square-free, then  $f$  is said to have a square-free value at  $u$ . Given a polynomial  $f \in \mathbb{F}_q[t][x]$ , we are motivated by the question of whether  $f$  has square-free values. Moreover, we would like to estimate the number of square-free values of  $f$  and show that it is large in some sense. If  $f$  is not square-free then we can not expect  $f$  to have many square-free values. This is because if  $g^2$  divides  $f$  where  $g \in \mathbb{F}_q[t][x]$  is a nonconstant polynomial, then for any  $u \in \mathbb{F}_q[t]$  such that  $\deg g(t, u(t)) > 0$ ,  $g(t, u(t))^2$  is a nonconstant square factor of  $f(t, u(t))$ .

Hence we require  $f$  to be a square-free polynomial. A natural question is then whether this condition is sufficient, i.e. whether a square-free polynomial always has square-free values.

This question may be viewed as a function field analogue of a known open conjecture which concerns polynomials over  $\mathbb{Z}$ . In the analogue question, instead of considering  $f$  as a polynomial over  $\mathbb{F}_q[t]$ ,  $f$  is considered as

a polynomial over  $\mathbb{Z}$ . The conjecture is that setting aside some simple exceptional cases, given a square-free polynomial  $f \in \mathbb{Z}[x]$  there are infinitely many  $n \in \mathbb{N}$  such that  $f(n)$  is a square-free number, and moreover, the set of square-free values of  $f$  has positive density.

The case where  $f$  is quadratic was solved by Ricci [14]. For the case where  $f$  is cubic, Erdős [3] showed that there are infinitely many square-free values, and Hooley [7] showed that the set of square-free values has positive density. Granville [4] showed that assuming the ABC conjecture the problem is completely settled.

Returning to the question over function fields, a quantitative statement of the question is to estimate the number of polynomials  $u \in \mathbb{F}_q[t]$  such that  $f(t, u(t))$  is square-free. This can be asked in the context of two limits. One is to fix a polynomial  $f$  and count number of  $u \in \mathbb{F}_q[t]$  of degree  $m$  such that  $f(t, u(t))$  is square-free while  $m$  tends to infinity. The other limit is to fix  $m$  and count the number of  $u \in \mathbb{F}_q[t]$  of degree  $m$  such that  $f(t, u(t))$  is square-free while  $q$  tends to infinity.

For any field  $\mathbb{F}$ , let

$$(1.1) \quad \mathcal{M}_m(\mathbb{F}) = \{u \in \mathbb{F}[t] : \deg u = m, u \text{ monic}\},$$

so that  $\#\mathcal{M}_m(\mathbb{F}_q) = q^m$ . Defining

$$(1.2) \quad \mathcal{S}_{\mathbb{F}}(f) = \{u \in \mathbb{F}[t] : f(t, u(t)) \text{ is square-free}\},$$

in [16] Rudnick studied the frequency

$$(1.3) \quad \frac{\#(\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{M}_m(\mathbb{F}_q))}{\#\mathcal{M}_m(\mathbb{F}_q)}$$

and showed that, assuming  $f \in \mathbb{F}_q[t][x]$  is separable with square-free content, as  $q \rightarrow \infty$ ,

$$(1.4) \quad \frac{(\#\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{M}_m(\mathbb{F}_q))}{\#\mathcal{M}_m(\mathbb{F}_q)} = 1 + O\left(\frac{(m \deg_x f + \deg_t f) \deg_x f}{q}\right),$$

where the implied constant is absolute. In the estimate above  $f$  is not assumed to be fixed. Indeed, fixing  $f$  makes little sense as the base field of  $\mathbb{F}_q$  may change as  $q \rightarrow \infty$ . However the estimate depends only on  $m$  and a bound on the degree of  $f$ , so  $f$  may vary while  $q \rightarrow \infty$  as long as its degree remains bounded.

In particular, Eq. 1.4 shows that if  $q$  is sufficiently large w.r.t.  $m$ ,  $\deg_x f$  and  $\deg_t f$ , then there exists  $u \in \mathcal{M}_m(\mathbb{F}_q)$  such that  $f(t, u(t))$  is square-free. Moreover, Eq. 1.4 shows that in some sense this is true for almost all  $u \in \mathcal{M}_m(\mathbb{F}_q)$ .

The key tool in [16] is the use of the discriminant of  $f(t, u(t))$  in order to tell whether  $f(t, u(t))$  is square-free. If  $f(t, u(t))$  is not square-free, the discriminant of  $f(t, u(t))$  vanishes. The last can be translated into a polynomial condition on the coefficients of  $u$ . Hence the problem can be converted to an algebraic statement about the number of zeros of a polynomial. It

may be interesting to note that this tool seems unavailable in the analogue question over  $\mathbb{Z}$ .

In this note we extend the results of Rudnick by considering a stronger version of the question. Instead of asking whether there exists a polynomial  $u \in \mathbb{F}_q[t]$  such that  $f(t, u(t))$  is square-free where  $u$  is a monic polynomial of degree  $m$ , we will ask whether there exists such polynomial  $u$  of a specific form, for example  $u = t^m + \beta$  where  $\beta \in \mathbb{F}_q$ . Throughout this note, when saying that a polynomial  $\tilde{u}$  is obtained by perturbing one or more coefficients of a polynomial  $u$ , we mean that  $\tilde{u}$  is obtained by changing only those coefficients of  $u$  while leaving the other coefficients of  $u$  unchanged. For example,  $t^m + 1$  is obtained by perturbing the free coefficient of  $t^m$ . Let  $\kappa \in \mathbb{N}$ , such that  $1 \leq \kappa \leq m$  and  $\kappa \not\equiv 0 \pmod{p}$ . Consider an arbitrary polynomial  $u \in \mathbb{F}_q[t]$ ,  $u(t) = \sum_{i=1, i \neq \kappa}^m \gamma_i t^i$ , where  $\gamma_1, \dots, \gamma_{\kappa-1}, \gamma_{\kappa+1}, \dots, \gamma_m \in \mathbb{F}_q$ . We will show that as  $q \rightarrow \infty$ , for almost all  $\gamma_0, \gamma_\kappa \in \mathbb{F}_q$ ,  $f(t, \sum_{i=0}^m \gamma_i t^i)$  is square-free. Namely, by perturbing two of the coefficients of  $u$  we obtain square-free values of  $f$ . The last is a special case of the main theorem of this note, in which we also consider similar sparse sets, more general than the set which corresponds to perturbations of two of the coefficients of a polynomial  $u$ .

As in [16] we do a similar use of the discriminant in order to translate the problem to an algebraic theorem which holds for any field. In Section 3 we describe how the discriminant may be used in showing the existence of square-free values. In this section we discuss first the case of assigning constants from the base field. This is the case where  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  is a multivariate polynomial, and we ask whether there exist  $\beta_1, \dots, \beta_d \in \mathbb{F}$  such that  $f(t, \beta_1, \dots, \beta_d)$  is square-free. In Section 3 we also extend the use of the discriminant properties, and in particular the fact that the expression for the discriminant is independent of the base field. By that we prove an algebraic lemma which holds over a general field  $\mathbb{F}$  in the case of constants assignments. The algebraic lemma which we present for constant assignments will also be used later, when we handle non-constant assignments.

The main result we show, provides an estimate of the number of square-free values of  $f$  in sparse subsets of  $\mathbb{F}_q[t]$  which are given by a linear condition of a certain kind. We now describe what these sparse sets are, and introduce the notations we use for defining them.

Let  $\mathbb{F}$  be a field. Let  $a, b, c \in \mathbb{F}[t]$ . Define

$$\mathcal{P}_{\mathbb{F}}(a, b, c) := \{a\beta_1 + b\beta_2 + c : \beta_1, \beta_2 \in \mathbb{F}\}.$$

For example, if  $c(t) = \sum_{i=0}^m \gamma_i t^i$  where  $\gamma_1, \dots, \gamma_m \in \mathbb{F}$ , then

$$\mathcal{P}_{\mathbb{F}}(1, 0, c) = \left\{ \sum_{i=1}^m \gamma_i t^i + \beta_1 : \beta_1 \in \mathbb{F} \right\}.$$

In this case  $\mathcal{P}_{\mathbb{F}}(1, 0, c)$  is the set of all polynomials in  $\mathbb{F}[t]$  that one gets by perturbing the free coefficient of  $c(t)$ . Similarly,  $\mathcal{P}_{\mathbb{F}}(1, t, c)$  denotes the set of

polynomials in  $\mathbb{F}[t]$  that one gets by perturbing the coefficient of  $t$  and the free coefficient of the polynomial  $c(t)$ .

$$\mathcal{P}_{\mathbb{F}}(1, t, c) = \left\{ \sum_{i=2}^m \gamma_i t^i + \beta_2 t + \beta_1 : \beta_1, \beta_2 \in \mathbb{F} \right\}$$

In general, if  $a, b \in \{1, t, t^2, \dots\}$  then  $\mathcal{P}_{\mathbb{F}}(a, b, c)$  denotes the polynomials obtained by perturbing two coefficients of  $c$ . If  $b = 0$  and  $a \in \{1, t, t^2, \dots\}$  then  $\mathcal{P}_{\mathbb{F}}(a, 0, c)$  corresponds to perturbing one coefficient of  $c(t)$ .

In the more general case where  $a, b$  are not necessarily in  $\{1, t, t^2, \dots\}$ ,  $\mathcal{P}_{\mathbb{F}}(a, b, c)$  is a subset of  $\mathbb{F}[t]$ . In the case of a finite field  $\mathbb{F}_q$ , the size of  $\mathcal{P}_{\mathbb{F}_q}(a, b, c)$  satisfies  $\#\mathcal{P}_{\mathbb{F}_q}(a, b, c) \leq q^2$ .

We are interested in finding condition on  $a$ ,  $b$  and  $c$  that guarantee the existence of square-free values of  $f$ , when  $u$  is restricted to the set  $\mathcal{P}_{\mathbb{F}_q}(a, b, c)$ , provided that  $q$  is sufficiently large. Moreover, we will see that for such  $a, b, c$ , as  $q \rightarrow \infty$   $f$  has a square-free value at almost all the elements of  $\mathcal{P}_{\mathbb{F}_q}(a, b, c)$ , that is:

$$\frac{\#(\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{P}_{\mathbb{F}_q}(a, b, c))}{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)} = 1 + O\left(\frac{1}{q}\right), \quad \text{as } q \rightarrow \infty.$$

We have

$$\frac{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)}{\#\mathcal{M}_m(\mathbb{F}_q)} \leq \frac{q^2}{q^m}.$$

Assuming  $m \geq 3$  and  $\deg a, \deg b, \deg c \leq m$ , then while keeping  $m$  fixed

$$\lim_{q \rightarrow \infty} \frac{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)}{\#\mathcal{M}_m(\mathbb{F}_q)} = 0.$$

This shows that if  $a, b, c \in \mathbb{F}_q[t]$  are such that  $\mathcal{P}_{\mathbb{F}_q}(a, b, c) \subseteq \mathcal{M}_m(\mathbb{F}_q)$ , then  $\mathcal{P}_{\mathbb{F}_q}(a, b, c)$  is sparse with respect to  $\mathcal{M}_m(\mathbb{F}_q)$  in the limit  $q \rightarrow \infty$ , so indeed claiming that there exists  $u \in \mathcal{P}_{\mathbb{F}_q}(a, b, c)$  such that  $f(t, u(t))$  is square-free for a given triple  $a, b, c \in \mathbb{F}_q[t]$  is stronger than claiming that there exists such  $u \in \mathcal{M}_m(\mathbb{F}_q)$ . We note that for some triples  $a, b, c$   $\mathcal{P}_{\mathbb{F}}(a, b, c)$  may not be a subset of  $\mathcal{M}_m(\mathbb{F}_q)$ . We allow such choice of  $a, b, c$  as well.

The main result in the case where  $f$  is a univariate polynomial over  $\mathbb{F}_q[t]$  is presented in Section 2, where we introduce the main results of this note. This result is proved in Section 5. In Section 6 we state and prove a generalization of this result to the case where  $f$  is a multivariate polynomial  $f \in \mathbb{F}[t][x_1, \dots, x_d]$ .

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### 1.1. Definitions and notations.

- (1)  $\mathbb{F}$  denotes a general field.  $\mathbb{F}_q$  denotes a finite field of  $q$  elements. The characteristic of  $\mathbb{F}$  is denoted by  $p$  or  $\text{Char}(\mathbb{F})$ . We also use  $L, K$  for general fields, in case that more than one field is considered.
- (2)  $\mathbb{F}[t][x]$  denotes the ring of polynomials in  $t$  and  $x$  over  $\mathbb{F}$ . By analogy with the ring of integers, we consider  $f$  as a univariate polynomial in  $x$  over the ring  $\mathbb{F}[t]$ , hence the notation. Similarly for multivariate polynomials over  $\mathbb{F}[t]$  we use the notation  $\mathbb{F}[t][x_1, \dots, x_d]$ .
- (3) Let  $D$  be a unique factorization domain. An element  $r \in D$  is **square-free** if every  $s \in D$  such that  $s^2 | r$  is invertible. Two elements  $v_1, v_2 \in D$  are called **associated** if there exists an invertible  $\alpha \in D$  such that  $v_1 = \alpha v_2$ . Let  $r = \prod_{i=1}^k r_i$  be a factorization of  $r$  into irreducible factors. Then  $r$  is square-free if and only if for every  $i, j$  such that  $i \neq j$ ,  $r_i$  and  $r_j$  are not associated. For our purposes  $D$  will be a polynomial ring. In cases where  $r$  can be considered as an element in two unique factorization domains  $D, \tilde{D}$  where  $\tilde{D} \supset D$ , we specify in which ring we assume  $r$  is square-free by saying that  $r$  is square-free in  $R$  or  $r$  square-free in  $\tilde{D}$ . The same meaning holds when saying that  $r$  is irreducible in  $D$ , or  $r$  is irreducible in  $\tilde{D}$ , and also when saying that  $d \in D$  divides  $r \in D$  in  $D$  or  $d$  divides  $r$  in  $\tilde{D}$ .
- (4) Let  $\mathbb{F}$  be a field - we denote by  $\overline{\mathbb{F}}$  an algebraic closure of  $\mathbb{F}$ , also  $\overline{\mathbb{F}(x)}$  denotes an algebraic closure of  $\mathbb{F}(x)$  etc. We also assume that  $\overline{\mathbb{F}(x)}$  is chosen such that it contains  $\overline{\mathbb{F}}$ .
- (5) For a vector  $(a_1, a_2, \dots, a_n) \in \mathbb{F}[t]^n$ , define

$$\|(a_1, \dots, a_n)\| := \max\{\deg a_1, \dots, \deg a_n\}.$$

- (6) A polynomial  $f \in \mathbb{F}[x]$  is **separable** if all its roots in an algebraic closure of  $\mathbb{F}$  are distinct. If  $f \in \mathbb{F}[x_1, \dots, x_d]$  is a multivariate polynomial, and  $i \in \mathbb{N}, 1 \leq i \leq d$ , then  $f$  is separable in  $x_i$  if  $f$  is separable when considering  $f$  as a univariate polynomial in the variable  $x_i$  over the field  $\mathbb{F}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$ .
- (7) Let  $D$  be an integral domain. A polynomial  $f \in D[x]$  is **primitive** if the only elements in  $D$  that divide all the coefficients of  $f$  are the invertible elements in  $D$ .
- (8) A field  $\mathbb{F}$  is **perfect** if either it has characteristic 0, or when  $p > 0$ , for any  $c \in \mathbb{F}$ ,  $c^{\frac{1}{p}} \in \mathbb{F}$  holds.
- (9) Let  $R_1, R_2$  be rings. Let  $R$  be a subring of  $R_1$  and  $R_2$ . A  **$R$ -homomorphism** is a homomorphism  $R_1 \rightarrow R_2$  such that  $r \mapsto r$  for every  $r \in R$ .
- (10) For a polynomial  $f \in \mathbb{F}[t][x_1, \dots, x_d]$ ,  $\deg_t f$  denotes the degree of  $f$  in the variable  $t$ , similarly, for  $i \in \mathbb{N}, 1 \leq i \leq d$ ,  $\deg_{x_i} f$  denotes the degree of  $f$  in  $x_i$ .  $\deg f$  denotes the total degree of  $f$  in all variables  $t, x_1, \dots, x_d$ .  $\deg_{\vec{x}} f$  denotes the total degree of  $f$  when considered as a polynomial in the variables  $x_1, \dots, x_d$  over  $\mathbb{F}[t]$ .

- (11) Let  $f \in \mathbb{F}[x]$ .  $\Delta f$  denotes the discriminant of  $f$ . Let  $\gamma_k \in \mathbb{F}$  be the leading coefficient of  $f$ . Then  $\Delta f = \gamma_k^{2k-2} \prod_{i < j} (r_i - r_j)^2$  where  $r_1, \dots, r_k$  are the roots of  $f$  in  $\overline{\mathbb{F}}$ .  $D^k$  denotes the expression for the discriminant in terms of the coefficients of  $f$ . For example, if  $f = \gamma_2 x^2 + \gamma_1 x + \gamma_0$  then  $D^k(f) = \gamma_1^2 - 4\gamma_2\gamma_0$ . If  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  is a multivariate polynomial, then  $\Delta_t, \Delta_{x_i}$  and  $D_t^k, D_{x_i}^k$  denote the corresponding notations when considering  $f$  as a univariate polynomial in  $t$  or  $x_i$  respectively.
- (12) Let  $a, b, c \in \mathbb{F}[t]$ . Let

$$\mathcal{P}_{\mathbb{F}}(a, b, c) := \{a\beta_1 + b\beta_2 + c : \beta_1, \beta_2 \in \mathbb{F}\}.$$

- (13) Given a polynomial  $f \in \mathbb{F}[t][x]$ , let

$$\mathcal{S}_{\mathbb{F}}(f) := \{u \in \mathbb{F}[t] : f(t, u(t)) \text{ is square-free}\}.$$

For a multivariate polynomial  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  the corresponding notation is

$$\mathcal{S}_{\mathbb{F},d}(f) := \{\vec{u} \in \mathbb{F}[t]^d : f(t, u_1(t), \dots, u_d(t)) \text{ is square-free}\}.$$

- (14) We denote the set of monic polynomials of degree  $m$  by  $\mathcal{M}_m(\mathbb{F})$ , namely  $\mathcal{M}_m(\mathbb{F}) = \{u \in \mathbb{F}[t] : \deg u = m, u \text{ monic}\}$ . In the case where  $\mathbb{F} = \mathbb{F}_q$  we abbreviate and write  $\mathcal{M}_m$ .

## 2. THE MAIN RESULTS

**2.1. Square-free values of a univariate polynomial.** We start by stating the main theorem for univariate polynomials in its general form, and then showing a few specific examples which are special cases of the general theorem. Recall that for polynomials  $a, b, c \in \mathbb{F}[t]$ , we define

$$\|(a, b, c)\| = \max\{\deg a, \deg b, \deg c\}.$$

**Theorem 2.1.** *Let  $f \in \mathbb{F}_q[t][x]$  be a square-free polynomial. Let  $a, b, c \in \mathbb{F}_q[t]$  such that  $\gcd(a, b) = 1$ . Let  $N \in \mathbb{N}$ . Assume  $\deg_x f, \deg_t f, \|(a, b, c)\| \leq N$ . Assume that at least one of the following holds*

- (1)  $p > C(N)$  where  $C(N)$  is a constant which depends only on  $N$ .
- (2)  $\frac{b}{a} \notin \mathbb{F}(t^p)$  where  $a \neq 0$ .

*Then while  $N$  remains fixed, we have:*

$$(2.1) \quad \frac{\#(\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{P}_{\mathbb{F}_q}(a, b, c))}{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)} = 1 + O\left(\frac{1}{q}\right), \quad \text{as } q \rightarrow \infty.$$

*In particular, if  $q$  is sufficiently large with respect to  $N$  then there exist  $\beta_1, \beta_2 \in \mathbb{F}_q$  such that  $f(t, c(t) + a(t)\beta_1 + b(t)\beta_2)$  is square-free.*

If  $q$  is taken to be large then  $p = \text{Char}(\mathbb{F}_q)$  may still remain small. For example if we fix a prime number  $p$ , then  $\mathbb{F}_q$  may be some algebraic extension of  $\mathbb{F}_p$  of large degree. On the other hand,  $p$  and  $q$  may both be large, for example if we take  $q = p$  and consider  $\mathbb{F}_p$  where  $p \rightarrow \infty$ . (1) in Theorem 2.1 can be viewed as the case where  $\text{Char}(\mathbb{F}_q)$  is large. Considering  $\mathbb{F}_p$  where

$p \rightarrow \infty$  is an example of this case. (2) provides the conditions on  $a, b$  in the case where  $\mathbb{F}_q$  is a field with an arbitrary positive characteristic. We introduce two examples of Theorem 2.1, one for each of the two cases.

**Example 2.2.** Let  $f \in \mathbb{F}_q[t][x]$  be square-free. Let  $c \in \mathbb{F}_q[t]$ , given by  $c(t) = \sum_{i=0}^m \gamma_i t^i$ . If we take  $b = 0, a = 1$  then  $\gcd(b, a) = 1$ . Hence by Theorem 2.1 if  $q$  and  $p$  are sufficiently large with respect to  $N$ , then there exists  $\beta_1 \in \mathbb{F}_q$  such that  $f(t, c(t) + \beta_1)$  is square-free, where  $c(t) + \beta_1$  is a polynomial obtained by a perturbation of the free coefficient of  $c$ .

**Example 2.3.** Let  $f \in \mathbb{F}_q[t][x]$  be square-free. Let  $\kappa \in \mathbb{N}$  such that  $\kappa \not\equiv 0 \pmod{p}$ . Let  $c \in \mathbb{F}_q[t]$ , given by  $c(t) = \sum_{i=0}^m \gamma_i t^i$ . If we take  $b = t^\kappa, a = 1$  then  $\gcd(b, a) = 1$ . Also  $\frac{b}{a} = t^\kappa \notin \mathbb{F}_q(t^p)$ . This shows that (2) in Theorem 2.1 holds. Hence by the same theorem if  $q$  is sufficiently large with respect to  $N$ , then there exist  $\beta_1, \beta_2$  such that  $f(t, c(t) + \beta_2 t^\kappa + \beta_1)$  is square-free, where  $c(t) + \beta_2 t^\kappa + \beta_1$  is a polynomial obtained by a perturbation the free coefficient and the coefficient of  $t^\kappa$  of  $c$ . In particular, in the case where  $\kappa = 1$ , a square-free value of  $f$  is obtained by perturbing the first two coefficients of  $c$ .

As the first example above shows, the large characteristic case allows us to take one of  $a$  or  $b$  to be 0, while the other be 1. This is because  $\gcd(1, 0) = 1$ , hence (1) of Theorem 2.1 holds for this choice of  $a, b$ . However, for an arbitrary positive characteristic both  $a$  and  $b$  are non-zero as this is required in (2) of Theorem 2.1. Hence in the case of a large characteristic it is sufficient to perturb a single coefficient of  $c$  in order to obtain a square-free value of  $f$ , while in the case of an arbitrary positive characteristic it might be necessary to perturb two coefficients of  $c$ .

The following two examples show why the assumption that  $\gcd(a, b) = 1$  is required in Theorem 2.1, and why the assumption that  $\frac{b}{a} \notin \mathbb{F}_q(t^p)$  is required in (2) of Theorem 2.1.

**Example 2.4.** Let  $a = t, b = t^2, c = 0$ . In this case  $\gcd(a, b) = t$ . Let  $f = x(x + t)$ . Then  $f$  is square-free but

$$f(t, t\beta_1 + t^2\beta_2) = (t\beta_1 + t^2\beta_2)(t\beta_1 + t^2\beta_2 + t) = t^2(\beta_1 + t\beta_2)(\beta_1 + t\beta_2 + 1)$$

which is divisible by  $t^2$ . Hence  $f(t, t\beta_1 + t^2\beta_2)$  is not square-free for any choice of  $\beta_1, \beta_2 \in \mathbb{F}_q$ .

**Example 2.5.** Let  $a = 1, b = t^p, c = t$ . Let  $f = x - t$ , which is irreducible and in particular square-free but

$$f(t, \beta_1 + t^p\beta_2 + t) = \beta_1 + t^p\beta_2 = \left( \beta_1^{\frac{1}{p}} + t\beta_2^{\frac{1}{p}} \right)^p.$$

Since  $\mathbb{F}_q$  is perfect  $\beta_1^{\frac{1}{p}}, \beta_2^{\frac{1}{p}} \in \mathbb{F}_q$ . Hence  $f(t, \beta_1 + t^p\beta_2 + t)$  is not square-free for any  $\beta_1, \beta_2 \in \mathbb{F}_q$ .

**Remark 1.** Let  $a, b, c \in \mathbb{F}_q[t]$  such that  $\gcd(a, b) \neq 1$ . By claiming that the condition  $\gcd(a, b) = 1$  is required in Theorem 2.1 we do not mean that Eq. 2.1 in Theorem 2.1 cannot hold for a specific choice of a square-free  $f$ .

Instead, by claiming that the condition is required we mean that if

$$\gcd(a, b) \neq 1$$

then there exists a square-free polynomial  $f$  such that  $f(t, \beta_1 a(t) + \beta_2 b(t) + c(t))$  is not square-free for any  $\beta_1, \beta_2 \in \mathbb{F}_q$ .

The same meaning applies also when we claim that the condition  $\frac{b}{a} \notin \mathbb{F}_q(t^p)$  is required in (2) of Theorem 2.1.

We do not place any restrictions on  $f$  in Theorem 2.1 other than that it should be square-free and that its degree remains bounded while  $q \rightarrow \infty$ . The conditions insure the existence of square-free values for any such  $f$ .

**2.2. Square-free values of a multivariate polynomial.** In Section 6 we state and prove Theorem 6.4 which is a generalization of Theorem 2.1 that holds for multivariate polynomials.

We use Theorem 6.4 in order to estimate the number of square-free values of a multivariate polynomial  $f$  at the set  $\mathcal{M}_{m_1} \times \cdots \times \mathcal{M}_{m_d}$  where  $m_1, \dots, m_d \in \mathbb{N}$ ,  $\deg_t f, \deg_{\bar{x}} f$  are fixed and  $q \rightarrow \infty$ . This result is stated in Corollary 2.6. Corollary 2.6 generalizes the result in [16] to the case of multivariate polynomials. In Section 6 we will show that it follows from Theorem 6.4.

**Corollary 2.6** (square-free values of multivariate polynomials over a finite field). *Let  $f \in \mathbb{F}_q[t][x_1, \dots, x_d]$  be a square-free polynomial. Let  $m_1, \dots, m_d \in \mathbb{N}$ . Let  $N \in \mathbb{N}$ . Assume  $\deg_{\bar{x}} f, \deg_t f, m_1, \dots, m_d \leq N$  and  $2 \leq m_1, \dots, m_d$ . Then while  $N$  remains fixed, the following holds:*

$$(2.2) \quad \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap (\mathcal{M}_{m_1} \times \cdots \times \mathcal{M}_{m_d}))}{\#\mathcal{M}_{m_1} \times \cdots \times \mathcal{M}_{m_d}} = 1 + O\left(\frac{1}{q}\right), \quad \text{as } q \rightarrow \infty.$$

*In particular, if  $q$  is sufficiently large with respect to  $N$  there exist  $u_1 \in \mathcal{M}_{m_1}, \dots, u_d \in \mathcal{M}_{m_d}$  such that  $f(t, u_1(t), \dots, u_d(t))$  is square-free.*

An estimate in the case where  $q$  is fixed and the degrees of  $u_1, \dots, u_d$  are allowed to grow was proved by Poonen in [12]. Let  $f \in \mathbb{F}_q[t][x_1, \dots, x_d]$  be a polynomial which is square-free as an element of  $K[x_1, \dots, x_d]$ , where  $K$  denotes the field of fractions of  $\mathbb{F}_q[t]$ . Let  $B_1, \dots, B_d \in \mathbb{N}$  and define

$$\text{Box} = \text{Box}(B_1, \dots, B_d) := \{(u_1, \dots, u_d) \in \mathbb{F}_q[t]^d : \deg u_i \leq B_i \text{ for all } i\}.$$

For a prime  $\mathfrak{p}$  in  $\mathbb{F}_q[t]$  let  $c_{\mathfrak{p}}$  denote then number of  $x \in (\mathbb{F}_q[t]/\mathfrak{p}^2)^d$  satisfying  $f(x) = 0$  in  $\mathbb{F}_q[t]/\mathfrak{p}^2$ . Poonen showed that

$$\lim_{B_1, \dots, B_d \rightarrow \infty} \frac{\#(\text{Box} \cap \mathcal{S}_{\mathbb{F}_q, d}(f))}{\#\text{Box}} = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{c_{\mathfrak{p}}}{|\mathfrak{p}|^{2d}}\right).$$



Theorem 6.4, which generalizes Theorem 2.1 to multivariate polynomials, will be stated in Section 6. Here we only introduce an example which is a specific case of Theorem 6.4.

**Example 2.7.** Let  $f \in \mathbb{F}_q[t][x_1, \dots, x_d]$  be a square-free polynomial. Let  $d > 0$  and let  $c_1, c_2, \dots, c_d \in \mathbb{F}_q[t]$ . In this example we perturb two coefficients of each of the polynomials  $c_1, c_2, \dots, c_d$  in order to obtain a square-free value of  $f$ . In the case where  $d = 1$  this example is the same as Example 2.3. Let  $\kappa_1, \kappa_2, \dots, \kappa_d \in \mathbb{N}$  such that  $\kappa_i \not\equiv 0 \pmod{p}$  for any  $i$ ,  $1 \leq i \leq d$ . Then if  $q$  is sufficiently large with respect to  $\deg_{\bar{x}} f, \deg_t f$  and  $\deg c_i, \kappa_i$  for  $1 \leq i \leq d$ , then there exist  $\beta_1, \dots, \beta_{2d} \in \mathbb{F}_q$  such that

$$(2.3) \quad f(t, \beta_1 + t^{\kappa_1} \beta_{d+1} + c_1, \beta_2 + t^{\kappa_2} \beta_{d+2} + c_2, \dots, \beta_d + t^{\kappa_d} \beta_{2d} + c_d)$$

is square-free. In particular, if  $\kappa_i = 1$ ,  $\forall i$ ,  $1 \leq i \leq d$ , a square-free value of  $f$  is obtained by perturbing the first two coefficients of  $c_1, \dots, c_d$ .

### 3. THE DISCRIMINANT AND CONSTANT ASSIGNMENTS OVER A GENERAL FIELD

In this section we work over a general field  $\mathbb{F}$  which is not necessarily finite. Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$ . We first consider a special case of the main question we are concerned with, that of substituting  $x_1, \dots, x_d$  with constants. By that we mean, we consider  $f(t, \beta_1, \dots, \beta_d) \in \mathbb{F}[t]$  where  $\beta_1, \dots, \beta_d \in \mathbb{F}$ .

In the special case of constant assignments, the main question we are concerned with, is to infer from the assumption that  $f$  is square-free, that an assignment  $f(t, \beta_1, \dots, \beta_d)$  is square-free. Instead of drawing such a connection between  $f$  being square-free to  $f(t, \beta_1, \dots, \beta_d)$  being square-free, Lemma 3.2, which is the main lemma of this section, shows a connection between  $f$  being separable in  $t$  to  $f(t, \beta_1, \dots, \beta_d)$  being separable in  $t$ . This connection is given by the existence of a polynomial  $P \in \mathbb{F}[x_1, \dots, x_d]$  which satisfies the following property: if  $f$  is separable in  $t$  then  $P$  is not the zero polynomial, while if  $f(t, \beta_1, \dots, \beta_d)$  is not separable in  $t$  then  $P(\beta_1, \dots, \beta_d)$  is zero. Since any non square-free polynomial in  $\mathbb{F}[t]$  is in particular not separable,  $P(\beta_1, \dots, \beta_d) = 0$  for any  $\beta_1, \dots, \beta_d$  such that  $f(t, \beta_1, \dots, \beta_d)$  is not square-free. Later this fact will be used in the proof of the main result.

In this section  $|$  denotes assignment. Hence

$$f|_{x_1=\beta_1, \dots, x_d=\beta_d}(t) = f(t, \beta_1, \dots, \beta_d) \in \mathbb{F}[t].$$

Also,  $\Delta$  denotes the discriminant of a polynomial as defined in Section 1.1.

The main observation, which is also the motivation for using the discriminant, is that  $f|_{x_1=\beta_1, \dots, x_d=\beta_d}$  has a multiple root in  $\overline{\mathbb{F}}$  if and only if  $\Delta(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = 0$ . Hence, the constant assignments  $\beta_1, \dots, \beta_d$  which result in a non separable polynomial  $f(t, \beta_1, \dots, \beta_d) \in \mathbb{F}[t]$ , can be identified as those that make the discriminant vanish. The last fact can be used in order to define the polynomial  $P$ , as we now show.

Let  $D^k \in \mathbb{Z}[x_0, \dots, x_k]$  be the polynomial which expresses the discriminant of a polynomial of degree  $k$  in terms of its coefficients. Let  $f \in \mathbb{F}[x]$

be a polynomial such that  $\deg f \leq k$ ,  $f = \sum_{i=0}^k \delta_i x^i$ . If  $\delta_k \neq 0$ , then the discriminant of  $f$  in terms of its coefficients is given by  $D^k$

$$\Delta f = D^k(\delta_0, \dots, \delta_k).$$

We use the notation  $D^k f := D^k(\delta_0, \dots, \delta_k)$ . For example if  $f(x) = \delta_2 x^2 + \delta_1 x + \delta_0$ , then  $D^k f = \delta_1^2 - 4\delta_2 \delta_0$ . We emphasize the distinction between  $\Delta f$  and  $D^k f$ . If  $\deg f = k$  then indeed  $\Delta f = D^k f$ . But if  $\deg f < k$  then this is not necessarily true.

If  $f \in \mathbb{F}[x_1, \dots, x_d]$  is a multivariate polynomial then the notation  $D_{x_i}^k$  will mean  $D^k f$  where  $f$  is viewed as a polynomial in variable  $x_i$  over

$$\mathbb{F}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d].$$

Namely, if

$$f(x_1, \dots, x_d) = \sum_{j=0}^k \delta_j(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) x_i^j$$

then

$$D_{x_i}^k(f) = D^k(\delta_0(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d), \dots, \delta_k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)).$$

In this case  $D_{x_i}^k(f) \in \mathbb{F}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d]$ .

The following fact, which we state as a lemma, is a direct consequence of the fact that  $D^k$  is a polynomial in  $\mathbb{Z}[x_0, \dots, x_k]$  which depends only on  $k$  and is the same regardless of the base field.

**Lemma 3.1.** *Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  where  $\deg_t f \leq k$ . Let  $\beta_1, \dots, \beta_d \in \mathbb{F}$ . Then*

$$(3.1) \quad (D_t^k f)|_{x_1=\beta_1, \dots, x_d=\beta_d} = D^k(f|_{x_1=\beta_1, \dots, x_d=\beta_d}).$$

The right hand side of Eq. 3.1 means first assigning  $x_1 = \beta_1, \dots, x_d = \beta_d$  to the polynomial  $f$ . The result is a polynomial in  $\mathbb{F}[t]$ . Then  $D^k$  is applied to the result. The left hand side of 3.1 means first applying  $D_t^k f$ . The result is a polynomial in  $\mathbb{F}[x_1, \dots, x_d]$ . Then assigning  $x_1 = \beta_1, \dots, x_d = \beta_d$  to the result. The order of operations is opposite in the two expressions. The lemma asserts that the two are equal. Let  $\mathcal{F}_k := \{f \in \mathbb{F}[t][x_1, \dots, x_d] : \deg_t f \leq k\}$ . Then the statement of the lemma is summarized by the commutative diagram below which holds for any polynomial  $f \in \mathcal{F}_k$ .

$$\begin{array}{ccc} \mathbb{F}[x_1, \dots, x_d] & \xleftarrow{D_t^k} & \mathcal{F}_k \\ \downarrow |_{x_1=\beta_1, \dots, x_d=\beta_d} & & \downarrow |_{x_1=\beta_1, \dots, x_d=\beta_d} \\ \mathbb{F} & \xleftarrow{D^k} & \mathbb{F}[t] \cap \mathcal{F}_k \end{array}$$

*Proof.*  $f(t, x_1, \dots, x_d) = \sum_{i=0}^k \delta_i(x_1, \dots, x_d) t^i$ . Hence

$$(3.2) \quad D_t^k f = D^k(\delta_0(x_1, \dots, x_d), \delta_1(x_1, \dots, x_d), \dots, \delta_k(x_1, \dots, x_d)).$$

Hence

$$(3.3) \quad (D_t^k f)|_{x_1=\beta_1, \dots, x_d=\beta_d} = D^k(\delta_0(\beta_1, \dots, \beta_d), \delta_1(\beta_1, \dots, \beta_d), \dots, \delta_k(\beta_1, \dots, \beta_d)).$$

Now,  $f|_{x_1=\beta_1, \dots, x_d=\beta_d} = f(t, \beta_1, \dots, \beta_d) = \sum_{i=0}^k \delta_i(\beta_1, \dots, \beta_d) t^i$ . Hence

$$(3.4) \quad D^k(f|_{x_1=\beta_1, \dots, x_k=\beta_k}) = D^k(\delta_0(\beta_1, \dots, \beta_k), \delta_1(\beta_1, \dots, \beta_k), \dots, \delta_k(\beta_1, \dots, \beta_k)).$$

As we mentioned before the proof,  $D^k$  in Eq. 3.3 and Eq. 3.4 is the same polynomial, although the base field is different, hence the expressions are equal.  $\square$

**Lemma 3.2.** *Let  $\mathbb{F}$  be a field. Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  be a polynomial. Then there exists a polynomial  $P \in \mathbb{F}[x_1, \dots, x_d]$  such that*

$$(3.5) \quad \{(\beta_1, \dots, \beta_d) \in \mathbb{F}^d : f(t, \beta_1, \dots, \beta_d) \text{ is not separable}\} \subseteq \{(\beta_1, \dots, \beta_d) \in \mathbb{F}^d : P(\beta_1, \dots, \beta_d) = 0\}$$

where

$$(3.6) \quad \deg P \leq (2 \deg_t f - 1) \deg_{\vec{x}} f.$$

The polynomial  $P$  is non-zero if and only if  $f$  is separable in  $t$ .

*Proof.* Let  $k = \deg_t f$ . Write  $f = \sum_{i=0}^k \delta_i(x_1, \dots, x_d) t^i$  where  $\delta_k \neq 0$ . Let  $P \in \mathbb{F}[x_1, \dots, x_d]$  defined by  $P := (D_t^k f) \cdot \delta_k$ . Note that  $\delta_k$  is nonzero, and  $D_t^k f = \Delta_t f$  is nonzero if and only if  $f$  is separable in  $t$ . Hence  $P$  is nonzero if and only if  $f$  is separable in  $t$ .

Now suppose  $\beta_1, \dots, \beta_d \in \mathbb{F}$  are such that  $f(t, \beta_1, \dots, \beta_d)$  is not separable. We need to show that  $P(\beta_1, \dots, \beta_d) = 0$ . Assume first  $\deg(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = k$ . Then  $\Delta(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = D^k(f|_{x_1=\beta_1, \dots, x_d=\beta_d})$ . Since  $f|_{x_1=\beta_1, \dots, x_d=\beta_d}$  is not separable,  $\Delta(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = 0$ . By Lemma 3.1 we get:

$$0 = \Delta(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = D^k(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) = (D_t^k f)|_{x_1=\beta_1, \dots, x_d=\beta_d}.$$

Hence  $(D_t^k f)(\beta_1, \dots, \beta_d) = 0$ . Now assume  $\deg(f|_{x_1=\beta_1, \dots, x_d=\beta_d}) < k$  then  $\delta_k(\beta_1, \dots, \beta_d) = 0$ . Hence in any case  $P(\beta_1, \dots, \beta_d) = 0$ .

It remains to bound the degree of  $P$ .

$$D_t^k f = D^k(\delta_0(x_1, \dots, x_d), \dots, \delta_k(x_1, \dots, x_d))$$

$$P = \delta_k D_t^k f = \delta_k(x_1, \dots, x_d) D^k(\delta_0(x_1, \dots, x_d), \dots, \delta_k(x_1, \dots, x_d))$$

$D^k \in \mathbb{Z}[x_0, \dots, x_k]$  is a homogenous polynomial of total degree  $2k - 2$ .  $\deg \delta_i \leq \deg_{\vec{x}} f$  for any  $i$ ,  $0 \leq i \leq k$ . We get

$$\deg(\delta_k D_t^k) \leq \deg_{\vec{x}} f + (2k - 2) \deg_{\vec{x}} f = (2k - 1) \deg_{\vec{x}} f = (2 \deg_t f - 1) \deg_{\vec{x}} f.$$

$\square$

#### 4. BACKGROUND AND GENERAL FACTS THAT ARE USED TO PROVE THE MAIN RESULTS

**4.1. Separable and square-free polynomials.** Since the following theorem and two consequences of it are not related to the main subject of the note, we state them in this section without a proof. As we did not find the exact theorems in another source, for the completeness of the note we prove them at the appendix.

**Theorem 4.1.** *Let  $\mathbb{F}$  be a field of positive characteristic  $p$ , and let  $\overline{\mathbb{F}}$  be the algebraic closure of  $\mathbb{F}$ . Let*

$$(4.1) \quad \mathbb{F}^{\frac{1}{p}} = \mathbb{F} \left( \left\{ c^{\frac{1}{p}} : c \in \mathbb{F} \right\} \right).$$

*Let  $f \in \mathbb{F}[x_1, \dots, x_d]$ . The following are equivalent:*

- (1)  *$f$  is square-free as an element in  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ .*
- (2)  *$f$  is square-free as an element in  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ .*
- (3)  *$f$  is square-free as an element in  $\mathbb{F}[x_1, \dots, x_d]$ , and  $f$  does not have an irreducible factor  $g$  such that  $g \in \mathbb{F}[x_1^p, \dots, x_d^p]$ .*

**Corollary 4.2.** *Let  $\mathbb{F}$  be a perfect field. Let  $\overline{\mathbb{F}}$  be the algebraic closure of  $\mathbb{F}$ . Let  $f \in \mathbb{F}[x_1, \dots, x_d]$ . Then  $f$  is square-free in  $\mathbb{F}[x_1, \dots, x_d]$  if and only if  $f$  is square-free in  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ .*

**Corollary 4.3.** *Let  $\mathbb{F}$  be a field. Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  be a square-free polynomial.*

- (1) *If  $\text{Char}(\mathbb{F}) = 0$  then  $f$  is separable in  $t$ .*
- (2) *If  $\text{Char}(\mathbb{F}) > 0$  and  $f$  is not separable in  $t$  then there exists an irreducible  $g \in \mathbb{F}[t^p][x_1, \dots, x_d]$  which divides  $f$ . Also  $\deg_t g > 0$ .*

**4.2. Properties of homomorphism.** The following two lemmas can be viewed as consequences of the structure preserving nature of homomorphisms.

**Lemma 4.4.** *Let  $D$  be an integral domain. Let  $\Psi : D \rightarrow D$  be a homomorphism. If  $u \in D$  is invertible then  $\Psi(u)$  is invertible.*

*Proof.* Since  $u$  is invertible, there exists  $u^{-1} \in D$  an inverse of  $u$ , and  $1 = \Psi(1) = \Psi(u)\Psi(u^{-1})$ .  $\square$

**Lemma 4.5.** *Let  $D$  be an integral domain. Let  $\Psi : D \rightarrow D$  be an automorphism. Let  $r \in D$ . Then  $r$  is irreducible if and only if  $\Psi(r)$  is irreducible.*

*Proof.* We show that if  $\Psi(r)$  is irreducible then  $r$  is irreducible. The opposite direction follows by symmetry when using the identity  $r = \Psi^{-1}(\Psi(r))$ . Suppose  $r = r_1 r_2$  where  $r_1, r_2 \in D$ . Then  $\Psi(r) = \Psi(r_1 r_2) = \Psi(r_1) \Psi(r_2)$ . Since  $\Psi(r)$  is irreducible one of  $\Psi(r_1), \Psi(r_2)$  must be invertible. Suppose without loss of generality that  $\Psi(r_2)$  is invertible. Hence by Lemma 4.4  $r_2 = \Psi^{-1}(\Psi(r_2))$  is invertible. Since  $r_1, r_2$  is an arbitrary factorization of  $r$  it follows that  $r$  is irreducible.  $\square$

**4.3. Derivation of rational functions.** Let  $R$  be a ring. An operation  $\delta : R \rightarrow R$  is called **derivation operator** if it satisfies the following two requirements for any two elements  $a, b \in R$ :

- (1)  $\delta(a + b) = \delta(a) + \delta(b)$ .
- (2)  $\delta(ab) = \delta(a)b + a\delta(b)$ .

For example,  $\frac{\partial}{\partial x_i}$  is a derivation operator of  $R[x_1, \dots, x_d]$ , as it satisfies both properties of a derivation.

Let  $R$  be a ring and let  $S$  be a multiplicative subset of  $R$ . That is,  $S$  is such that for any  $s_1, s_2 \in S$ ,  $s_1 s_2 \in S$ . By a standard construction there exists a ring which contains quotients  $\frac{a}{s}$  where  $a \in R$  and  $s \in S$ , which we denote by  $S^{-1}R$ . Any derivation operator of  $R$  can be extended to a derivation operator of  $S^{-1}R$ , where the derivation in  $S^{-1}R$  is given by the usual quotient rule for derivatives. We state this fact in the following proposition.

**Proposition 4.6.** *Let  $R$  be a ring, let  $\delta$  be a derivation operator of  $R$ , and let  $S$  be a multiplicative subset of  $R$ . Let  $r_1, r_2 \in R, s_1, s_2 \in S$  such that  $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ . Then*

$$\frac{\delta(r_1)s_1 - r_1\delta(s_1)}{s_1^2} = \frac{\delta(r_2)s_2 - r_2\delta(s_2)}{s_2^2}.$$

A proof of the above proposition and more information about the extension of the derivation operator to  $S^{-1}R$  can be found in [9] Chapter 1.

We are interested in the case where the ring is a polynomial ring over a field,  $\mathbb{F}[x_1, \dots, x_d]$ , and the derivation operators are given by the formal partial derivatives  $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}$ . We will also derive rational functions over a field, and the meaning of that is made precise by the general facts about derivations ring which are described above.

For convenience of reference, we state the following known fact about derivations which is used in this note.

**Proposition 4.7 (Chain Rule).** *Let  $f \in \mathbb{F}(x_1, \dots, x_d)$ , let  $g = (g_1, \dots, g_d)$  where  $g_1, \dots, g_d \in \mathbb{F}(x)$ , and let  $f \circ g \in \mathbb{F}(x)$ ,  $f \circ g(x) = f(g_1(x), \dots, g_d(x))$  then*

$$(4.2) \quad \frac{df \circ g}{dx} = \sum_{i=1}^d \frac{\partial f}{\partial x_i} \frac{dg_i}{dx}.$$

## 5. PROOF OF THEOREM 2.1

The main steps in the proof of Theorem 2.1 are as follows. First, we introduce Theorem 5.2, which is an algebraic theorem which holds for any field  $\mathbb{F}$ . Secondly, we show that Theorem 2.1 is a consequence of Theorem 5.2, when the latter is applied in the special case where  $\mathbb{F}$  is assumed to be a finite field. Finally, we prove Theorem 5.2. For that purpose, we apply Lemma 3.2 which was introduced in Section 3.

### 5.1. Reduction to an algebraic theorem which holds for any field.

The following lemma provides an elementary upper bound on the number of zeros of a multivariate polynomial over a finite field. A proof can be found in [17] Chapter 4.

**Lemma 5.1.** *Let  $P \in \mathbb{F}_q[x_1, \dots, x_d]$  be a non-zero polynomial of total degree  $n$ . Then the number  $\mathcal{N}_P$  of zeros of  $P(x_1, \dots, x_d)$  in  $\mathbb{F}_q^d$  satisfies*

$$(5.1) \quad \mathcal{N}_P \leq nq^{d-1}.$$

Theorem 2.1 is a consequence of the following theorem which holds for any field  $\mathbb{F}$ .

**Theorem 5.2.** *Let  $\mathbb{F}$  be a field, and let  $\overline{\mathbb{F}}$  be an algebraic closure of  $\mathbb{F}$ . Let  $f \in \mathbb{F}[t][x]$  be a polynomial which is square-free in  $\overline{\mathbb{F}}[t][x]$ . Let  $a, b, c \in \mathbb{F}[t]$  such that  $\gcd(a, b) = 1$ . Let  $N \in \mathbb{N}$ . Assume  $\deg_x f, \deg_t f, \|(a, b, c)\| \leq N$ . Then there exists a polynomial  $P_{f,a,b,c} \in \mathbb{F}[x_1, x_2]$  which depends on  $a, b, c$  and  $f$  such that*

$$(5.2) \quad \{(\beta_1, \beta_2) \in \mathbb{F}^2 : f(t, a(t)\beta_1 + b(t)\beta_2 + c(t)) \text{ is not separable}\} \\ \subseteq \{(\beta_1, \beta_2) \in \mathbb{F}^2 : P_{f,a,b,c}(\beta_1, \beta_2) = 0\}.$$

Moreover, there exists a constant  $\tilde{C}(N)$  which depends only on  $N$  such that

$$\deg P_{f,a,b,c} \leq \tilde{C}(N).$$

$P_{f,a,b,c}$  is non-zero if at least one of the following holds:

- (1)  $p = 0$  or  $p > C(N)$  where  $C(N)$  is a constant which depends only on  $N$ .
- (2)  $\frac{b}{a} \notin \mathbb{F}(t^p)$  where  $a \neq 0$ .

We now show why Theorem 2.1 follows from Theorem 5.2 in the case where  $\mathbb{F}$  is a finite field. First, over a finite field the requirement in Theorem 5.2 that  $f$  is square-free in  $\overline{\mathbb{F}}[t][x]$  can be replaced by the requirement that  $f$  is square-free. Over a finite field the two requirements are equivalent by Corollary 4.2, since  $\mathbb{F}_q$  is a perfect field. Likewise, by the same corollary over a finite field  $f(t, a(t)\beta_1 + b(t)\beta_2 + c(t))$  being separable and  $f(t, a(t)\beta_1 + b(t)\beta_2 + c(t))$  being square-free can be used interchangeably, since the two are equivalent over a perfect field.

With the assumptions and definitions as in Theorem 2.1, we now show why the estimate in Eq. 2.1 of Theorem 2.1 follows. Let  $\mathcal{S}_{\mathbb{F}_q}(f)^c$  be the complement of  $\mathcal{S}_{\mathbb{F}_q}(f)$  in  $\mathbb{F}_q[t]$ . The error  $E(q)$  of estimating

$$\frac{\#(\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{P}_{\mathbb{F}_q}(a, b, c))}{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)}$$

by 1 is given by

$$(5.3) \quad E(q) = 1 - \frac{\#(\mathcal{S}_{\mathbb{F}_q}(f) \cap \mathcal{P}_{\mathbb{F}_q}(a, b, c))}{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)} = \frac{\#(\mathcal{S}_{\mathbb{F}_q}(f)^c \cap \mathcal{P}_{\mathbb{F}_q}(a, b, c))}{\#\mathcal{P}_{\mathbb{F}_q}(a, b, c)} =$$

$$\frac{\#\{(\beta_1, \beta_2) \in \mathbb{F}_q^2 : f(t, a(t)\beta_1 + b(t)\beta_2 + c(t)) \text{ is not square-free}\}}{q^2} \leq$$

$$\frac{\#\{(\beta_1, \beta_2) \in \mathbb{F}_q^2 : P_{f,a,b,c}(\beta_1, \beta_2) = 0\}}{q^2}.$$

Assume first that  $a, b, c$  are such that  $P_{f,a,b,c}$  is nonzero. Then applying Lemma 5.1

$$E(q) \leq \frac{\tilde{C}(N)q}{q^2}.$$

Hence keeping  $N$  fixed while  $q \rightarrow \infty$ ,  $E(q) = O\left(\frac{1}{q}\right)$ .

It remains to show why it follows from the assumptions of Theorem 2.1 that  $P_{f,a,b,c}$  is non-zero. According to Theorem 5.2  $P_{f,a,b,c}$  is non-zero if at least one of (1) and (2) in the same theorem holds. Indeed, by letting  $C(N)$  in Theorem 2.1 be the same as  $C(N)$  in Theorem 5.2, (1) and (2) in Theorem 5.2 are the same as (1) and (2) in Theorem 2.1. Hence by the assumptions of Theorem 2.1 at least one of the two holds.

**Remark 2.** The statement that  $P_{f,a,b,c}$  is nonzero is crucial for the reduction, since this is required in order for the bound in Eq. 5.1 to hold. In the proof of Theorem 5.2 the existence of  $P_{f,a,b,c}$  will be provided by Lemma 3.2 which we proved in Section 3, and the part of Theorem 5.2 about  $P_{f,a,b,c}$  not being the zero polynomial will be given by the same lemma.

**5.2. A single coefficient perturbation.** We now turn to prove Theorem 5.2. To illustrate the main steps in the proof, we start by proving a special case of Theorem 5.2 where  $\text{Char}(\mathbb{F})$  is large, namely (1) in Theorem 5.2 holds,  $c \in \mathbb{F}[t]$  is an arbitrary polynomial,  $a = 1$  and  $b = 0$ , which corresponds to perturbations of the free coefficient of  $c$ . In the case where  $\mathbb{F}$  is a finite field, by the reduction of Section 5.1, this implies the special case of Theorem 2.1 which is given in Example 2.2 of Section 2.

**Definition 5.3.** Let  $R, R_1, R_2$  be rings such that  $R \subseteq R_1, R \subseteq R_2$ . A homomorphism  $\Psi : R_1 \rightarrow R_2$  is a  **$R$ -homomorphism** if for any  $r \in R$ ,  $\Psi(r) = r$ .

Let  $c \in \mathbb{F}[t]$ . There exist a unique  $\mathbb{F}[t]$ -automorphism  $\Psi : \mathbb{F}[t][x] \rightarrow \mathbb{F}[t][x]$  such that  $x \mapsto x + c$ . This automorphism is given by  $f(t, x) \mapsto f(t, x + c(t))$ .

**Lemma 5.4.** Let  $f \in \mathbb{F}[t][x]$  be a square-free polynomial, and let  $c \in \mathbb{F}[t]$ . Let  $\Psi : \mathbb{F}[t][x] \rightarrow \mathbb{F}[t][x]$  be the unique  $\mathbb{F}[t]$ -automorphism defined by  $x \mapsto x + c$ . If  $p = 0$  or  $p > \deg_x(f) \deg(c) + \deg_t(f)$  then  $\Psi(f)$  is separable in  $t$ .

*Proof.* Let  $f = \prod_{i=1}^k f_i$  be a factorization of  $f$  into irreducible factors. Then by homomorphism properties

$$(5.4) \quad \Psi(f) = \prod_{i=1}^k \Psi(f_i).$$

Since  $\Psi$  is an automorphism, by Lemma 4.5  $\Psi(f_1), \dots, \Psi(f_k)$  are irreducible. Hence Eq. 5.4 provides a factorization of  $\Psi(f)$  into irreducible factors in  $\mathbb{F}[t][x]$ . We now show that  $\Psi(f)$  is square-free. Suppose there exist  $f_i, f_j$  where  $i \neq j$  such that  $\Psi(f_i)$  and  $\Psi(f_j)$  are associated, that is  $\Psi(f_i) = \alpha \Psi(f_j)$  where  $\alpha \in \mathbb{F}$ . Since  $\Psi(\alpha) = \alpha$  we have  $\Psi(f_i) = \Psi(f_j)\Psi(\alpha) = \Psi(f_j\alpha)$ . But  $\Psi$  is injective, hence  $f_i = f_j\alpha$ . But that is a contradiction to the assumption that  $f$  is square-free. Hence  $\Psi(f)$  is square-free. If  $p = 0$  then we are done, since it follows by Corollary 4.3 part 1 that  $\Psi(f)$  is separable in  $t$ .

Now suppose  $p > 0$ , and suppose on the contrary that  $\Psi(f)$  is not separable as polynomial in  $t$ . By Corollary 4.3 part 2  $\Psi(f)$  has an irreducible factor in  $\mathbb{F}[t^p][x]$ , where its degree in  $t$  is not zero. Without loss of generality, we can assume this irreducible factor is  $\Psi(f_i)$ , for some  $i$ ,  $1 \leq i \leq k$ . But the latter cannot hold since for any  $i$ ,  $1 \leq i \leq k$

$$\deg_t(\Psi(f_i)) \leq \deg_t(\Psi(f)) \leq \deg(c) \deg_x(f) + \deg_t(f) < p,$$

where the last inequality is by our assumption on  $p$ .  $\square$

We now prove Theorem 5.2 for the special case of a single variable perturbation, where we assume (1) of Theorem 5.2 holds.

*Proof.* Let  $\Psi : \mathbb{F}[t][x] \rightarrow \mathbb{F}[t][x]$  be the unique  $\mathbb{F}[t]$ -automorphism which is given by  $x \mapsto c$ . Let  $\tilde{f} := \Psi(f)$ . The proof will follow by applying Lemma 3.2 to  $\tilde{f}$ . Let  $C(N)$  be sufficiently large such that  $C(N) \geq \deg(c) \deg_x(f) + \deg_t(f)$  holds. Since we assume that (1) in Theorem 5.2 holds, by Lemma 5.4,  $\tilde{f}$  is separable as polynomial in variable  $t$ . Hence by Lemma 3.2 there exists a non zero  $P \in \mathbb{F}[x]$  such that

$$\{\beta \in \mathbb{F} : \tilde{f}(t, \beta) \text{ is not separable}\} \subseteq \{\beta \in \mathbb{F} : P(\beta) = 0\}.$$

But

$$\{\beta \in \mathbb{F} : \tilde{f}(t, \beta) \text{ is not separable}\} = \{\beta \in \mathbb{F} : f(t, \beta + c(t)) \text{ is not separable}\}$$

Also,

$$\deg_t \tilde{f} \leq \deg_x f \deg c + \deg_t f$$

and

$$\deg_x \tilde{f} = \deg_x f.$$

Assigning this into the bound which is given by Eq. 3.6 in Lemma 3.2 we get

$$\deg P \leq (2(\deg_x f \deg c + \deg_t f) - 1) \deg_x f.$$

Let  $\tilde{C}(N)$  be such that  $\tilde{C}(N) \geq (2(\deg_x f \deg c + \deg_t f) - 1) \deg_x f$ , then the bound on  $\deg P$  holds as required.  $\square$



**5.3. Proof of Theorem 5.2.** The proof of Theorem 5.2 follows similar lines as the proof of a special case of the same theorem in the previous section, only that now we consider more homomorphisms of  $\mathbb{F}[t][x]$  other than the one which maps  $x$  to  $x + c$  where  $c \in \mathbb{F}[t]$ . Also, in Lemma 5.4 and its proof we assumed  $p$  is sufficiently large. Lemma 5.7 which we prove in this section generalizes Lemma 5.4 and its proof to the case of an arbitrary characteristic  $p$ .

Let  $R$  be a ring. Every  $R$ -homomorphism  $R[x_1] \rightarrow R[x_1, x_2]$  is uniquely determined by the image of  $x_1$ . Conversely, for any  $f \in R[x_1, x_2]$  there is an  $R$ -homomorphism  $R[x_1] \rightarrow R[x_1, x_2]$  such that  $x_1 \mapsto f$ . This homomorphism is given by  $g \mapsto g \circ f$  for any  $g \in R[x_1]$ . We denote by  $\Psi_f$  the unique  $R$ -homomorphism  $\Psi_f : R[x_1] \rightarrow R[x_1, x_2]$  such that  $x_1 \mapsto f$ .

In the following lemma we assume  $D$  is an integral domain. For our purposes, we need only the case where  $D = \mathbb{F}[t]$ .

**Lemma 5.5.** *Let  $D$  be an integral domain. Let  $a, b, c \in D$ , such that  $a \neq 0$  and  $l \in D[x_1, x_2]$ ,  $l(x_1, x_2) := ax_1 + bx_2 + c$ . Let  $\Psi_l : D[x_1] \rightarrow D[x_1, x_2]$  be a  $D$ -homomorphism defined by  $x_1 \mapsto l$ .*

- (1) *Let  $K$  be the field of fractions of  $D$ .  $\Psi_l$  can be extended to an automorphism  $\tilde{\Psi} : K[x_1, x_2] \rightarrow K[x_1, x_2]$ . In particular,  $\Psi_l$  is injective.*
- (2) *If  $D$  is a field, and if  $f$  is irreducible in  $D[x_1]$  then  $\Psi_l(f)$  is irreducible in  $D[x_1, x_2]$ .*
- (3) *If  $D$  is a unique factorization domain,  $\gcd(a, b) = 1$  and  $f$  is primitive in  $D[x_1]$  then the greatest common divisor of the coefficients of  $\Psi_l(f)$  in  $D$  is 1.*
- (4) *If  $D$  is a unique factorization domain,  $\gcd(a, b) = 1$  and  $f$  is irreducible in  $D[x_1]$  then  $\Psi_l(f)$  is irreducible in  $D[x_1, x_2]$ .*

*Proof.* 1:  $\Psi_l$  can be extended to a  $K$ -homomorphism  $\tilde{\Psi} : K[x_1, x_2] \rightarrow K[x_1, x_2]$  by the following rule

$$x_1 \mapsto ax_1 + bx_2 + c,$$

$$x_2 \mapsto x_2.$$

$\tilde{\Psi}$  is an automorphism, with an inverse homomorphism  $\tilde{\Psi}^{-1} : K[x_1, x_2] \rightarrow K[x_1, x_2]$  which is given by

$$x_1 \mapsto \frac{1}{a}(x_1 - bx_2 - c)$$

$$x_2 \mapsto x_2.$$

Hence the claim follows.

2: It follows from the assumption that  $f$  is irreducible in  $D[x_1]$ , that  $f$  is irreducible also in  $D[x_1, x_2]$ . This is because for every factorization of  $f$ ,  $f = g_1 g_2$  where  $g_1, g_2 \in D[x_1, x_2]$ ,  $\deg_{x_2} g_1 = \deg_{x_2} g_2 = \deg_{x_2} f = 0$ . Hence  $g_1, g_2 \in D[x_1]$ . By part 1 of the lemma, since  $D = K$  in this case,  $\Psi_l$  can be extended to an automorphism  $\tilde{\Psi}$  of  $D[x_1, x_2]$ . Since we assume  $f$  is

irreducible in  $D[x_1, x_2]$ , by Lemma 4.5  $\tilde{\Psi}(f)$  is irreducible in  $D[x_1, x_2]$ . But  $\Psi_l(f) = \tilde{\Psi}(f)$ . Hence  $\Psi_l(f)$  is irreducible in  $D[x_1, x_2]$ .

3: Let  $h$  be a prime element in  $D$ . Then  $\langle h \rangle$  is a prime ideal. Hence  $D/\langle h \rangle$  is an integral domain. Let  $\overline{D} = D/\langle h \rangle$ .

We now define few notations. For an element  $c \in D$  denote by  $\bar{c}$  the equivalence class of  $c$  in  $\overline{D}$ . For a polynomial  $g \in \mathbb{F}[x_1]$ ,  $g = \sum_{i=0}^n c_i x_1^i$  denote  $\bar{g} = \sum_{i=0}^n \bar{c}_i x_1^i$ . Likewise, for a polynomial  $g \in \mathbb{F}[x_1, x_2]$ ,  $g = \sum_{0 \leq i \leq n_1, 0 \leq j \leq n_2} c_{i,j} x_1^i x_2^j$  denote  $\bar{g} = \sum_{0 \leq i \leq n_1, 0 \leq j \leq n_2} \bar{c}_{i,j} x_1^i x_2^j$ .

Let  $\Psi_{\bar{l}} : \overline{D}[x_1] \rightarrow \overline{D}[x_1, x_2]$  be a  $\overline{D}$ -homomorphism defined by  $x \mapsto \bar{l}$ . Since  $\gcd(a, b) = 1$ , at least one of  $\bar{a}, \bar{b}$  is nonzero. Hence by part 1  $\overline{D}$  can be extended to an automorphism  $\overline{K}[x_1, x_2] \rightarrow \overline{K}[x_1, x_2]$  where  $\overline{K}$  is the field of fractions of  $\overline{D}$ . In particular, the kernel of  $\Psi_{\bar{l}}$  is trivial. Since  $f$  is primitive, hence  $\bar{f} \neq 0 \pmod{\langle h \rangle}$ , it follows that  $\Psi_{\bar{l}}(\bar{f}) \neq 0 \pmod{\langle h \rangle}$ . But  $\overline{\Psi_l(f)} = \Psi_{\bar{l}}(\bar{f})$  by homomorphism properties. Hence  $h$  does not divide all coefficients of  $\Psi_l(f)$ . Since  $h$  is arbitrary the claim follows.

4: Let  $K$  be the field of fractions of  $D$ . Since  $f$  is irreducible in  $D[x_1]$ , by Gauss's lemma for polynomials it is irreducible in  $K[x_1]$  and primitive. By part 2 applied on  $f$  as an element in  $K[x_1]$ ,  $\Psi_l(f)$  is irreducible in  $K[x_1, x_2]$ . Since  $f$  is primitive, by part 3 the coefficients of  $\Psi_l(f)$  in  $D$  do not have a nontrivial common divisor. Since  $\Psi_l(f)$  is irreducible in  $K[x_1, x_2]$  and does not have a nontrivial factor in  $D$ , it is irreducible in  $D[x_1, x_2]$ .  $\square$

Although in the main theorem  $a, b, c$  and  $f$  are assumed to be polynomials, for the following lemma we assume  $a, b, c$  and  $f$  are rational functions and not necessarily polynomials, i.e.  $a, b, c \in \mathbb{F}(t)$  and  $f \in \mathbb{F}(t, x_1)$ . We will also perform formal derivations of functions in  $\mathbb{F}(t, x_1, x_2)$ . The meaning of that is described in Section 4.3.

**Lemma 5.6.** *Let  $\mathbb{F}$  be a field of positive characteristic  $p$ . Let  $f \in \mathbb{F}(t, x_1)$ . Let  $a, b, c \in \mathbb{F}(t)$  such that  $a \neq 0$  and  $\frac{b}{a} \notin \mathbb{F}(t^p)$ . Suppose  $f(t, a(t)x_1 + b(t)x_2 + c(t)) \in \mathbb{F}(t^p, x_1, x_2)$ . Then  $f \in \mathbb{F}(t^p, x_1^p)$ .*

In the proof of Lemma 5.6 we use the fact that  $f \in \mathbb{F}(t^p)$  if and only if  $\frac{df}{dt} = 0$ , which is a known property of derivations.

*Proof.* First we note that it is sufficient to prove the lemma in the case where  $a = 1$  and  $c = 0$ . This is because if we define  $\bar{f}(t, x_1) := f(t, a(t)x_1 + c(t))$ , and define  $\bar{a} := 1, \bar{b} := \frac{b}{a}, \bar{c} := 0$ , then  $f(t, a(t)x_1 + b(t)x_2 + c(t)) = \bar{f}(t, x_1 + \frac{b}{a}(t)x_2) = \bar{f}(t, \bar{a}(t)x_1 + \bar{b}(t)x_2 + \bar{c}(t))$ . Proving the lemma for the special case will show that if  $\bar{f}(t, x_1 + \frac{b}{a}(t)x_2) \in \mathbb{F}(t^p, x_1^p)$  then  $\bar{f} \in \mathbb{F}(t^p, x_1^p)$ , i.e.

$\bar{f} = g(t^p, x_1^p)$  for some  $g \in \mathbb{F}(t, x_1)$ . But then

$$(5.5) \quad f(t, x_1) = \bar{f}\left(t, \frac{x_1 - c(t)}{a(t)}\right) = g\left(t^p, \left(\frac{x_1 - c(t)}{a(t)}\right)^p\right) = g\left(t^p, \frac{x_1^p - c(t)^p}{a(t)^p}\right) \in \mathbb{F}(t^p, x_1^p).$$

Hence it is sufficient to prove the case where  $f = \bar{f}, a = \bar{a}, b = \bar{b}, c = \bar{c}$ . From now on we assume  $a = 1$  and  $c = 0$ .

Now, let  $\tilde{f} := f(t, x_1 + b(t)x_2)$ . Suppose  $\tilde{f} \in \mathbb{F}(t^p, x_1, x_2)$ . Then by the chain rule

$$0 = \frac{\partial \tilde{f}}{\partial t}(t, x_1, x_2) = \frac{\partial f}{\partial x_1}(t, x_1 + b(t)x_2)b'(t)x_2 + \frac{\partial f}{\partial t}(t, x_1 + b(t)x_2).$$

By a change of variables  $x_1 = x_1 - b(t)x_2$  we get:

$$0 = \frac{\partial f}{\partial x_1}(t, x_1)b'(t)x_2 + \frac{\partial f}{\partial t}(t, x_1).$$

We view the above as a polynomial in variable  $x_2$  over  $\mathbb{F}(t, x_1)$ . By equating the coefficients of this polynomial to 0 we get the following two equations

$$(5.6) \quad \frac{\partial f}{\partial t}(t, x_1) = 0$$

$$(5.7) \quad \frac{\partial f}{\partial x_1}(t, x_1)b'(t) = 0$$

Since  $b'(t) \neq 0$  by our assumption that  $\frac{b}{a} = b \notin \mathbb{F}(t^p)$ , Eq. 5.7 holds if and only if  $\frac{\partial f}{\partial x_1} = 0$ . Hence the two equations, Eq. 5.6 and Eq. 5.7, hold if and only if both derivatives by  $x_1$  and by  $t$  vanish. Equivalently,  $f \in \mathbb{F}(t^p, x_1^p)$ .  $\square$

**Lemma 5.7.** *Let  $f \in \mathbb{F}[t][x_1]$  be square-free in  $\bar{\mathbb{F}}[t][x_1]$ . Let  $a, b, c \in \mathbb{F}[t]$  such that  $\gcd(a, b) = 1$ . Let  $l \in \mathbb{F}[t][x_1, x_2]$  defined by  $l(t, x_1, x_2) := a(t)x_1 + b(t)x_2 + c(t)$ . Let  $\Psi_l : \mathbb{F}[t][x_1] \rightarrow \mathbb{F}[t][x_1, x_2]$  be the  $\mathbb{F}[t]$ -homomorphism defined by  $x_1 \mapsto l$ . If at least one of the following holds then  $\Psi_l(f)$  is separable in  $t$ .*

- (1)  $p = 0$  or  $p > \|(a, b, c)\| \deg_{x_1} f + \deg_t f$ .
- (2)  $\frac{b}{a} \notin \mathbb{F}(t^p)$  where  $a \neq 0$ .

*Proof.* Let  $f = \prod_{i=1}^k f_i$  be a factorization of  $f$  into irreducible factors. Then by homomorphism properties

$$(5.8) \quad \Psi_l(f) = \prod_{i=1}^k \Psi_l(f_i).$$

By Lemma 5.5 part 4 Eq. 5.8 provides a factorization of  $\Psi_l(f)$  into irreducible factors in  $\mathbb{F}[t][x_1, x_2]$ . We now show that  $\Psi_l(f)$  is square-free. Suppose  $\Psi_l(f_i) = \Psi_l(f_j)\alpha$  where  $i \neq j$  and  $\alpha \in \mathbb{F}$ . Since  $\Psi_l(\alpha) = \alpha$  we have

$\Psi_l(f_i) = \Psi_l(f_j)\Psi_l(\alpha) = \Psi_l(f_j\alpha)$ . Since  $\Psi_l$  is injective by Lemma 5.5 part 1, we conclude that  $f_i = f_j\alpha$ . But that is a contradiction to the assumption that  $f$  is square-free. Hence  $\Psi_l(f)$  is square-free. If  $p = 0$  then we are done, since by Corollary 4.3 part 1  $\Psi_l(f)$  being square-free implies that  $\Psi_l(f)$  is separable in  $t$ .

Now suppose  $p > 0$ , and suppose on the contrary that  $\Psi_l(f)$  is not separable as polynomial in  $t$ . Then by Corollary 4.3 part 2  $\Psi_l(f)$  has an irreducible factor in  $\mathbb{F}[t^p][x_1, x_2]$ , where its degree in  $t$  is not zero. Thus we can assume without loss of generality that for some  $i$ ,  $1 \leq i \leq k$

$$(5.9) \quad \Psi_l(f_i) \in \mathbb{F}[t^p][x_1, x_2], \text{ where } \deg_t \Psi_l(f_i) > 0.$$

We will now show that Eq. 5.9 cannot hold. The proof splits here, depending on which of (1) or (2) of the lemma holds.

Suppose (1) holds. Then Eq. 5.9 cannot hold since for any  $i$ ,  $1 \leq i \leq k$

$$\deg_t(\Psi_l(f_i)) \leq \deg_t(\Psi_l(f)) \leq \|(a, b, c)\| \deg_{x_1} f + \deg_t f < p$$

where the last inequality is by our assumption on  $p$ .

Now suppose (2) holds. Since  $\frac{b}{a} \notin \mathbb{F}(t^p)$ , by Lemma 5.6  $f_i \in \mathbb{F}[t^p, x_1^p]$ . Now by the equivalent conditions in Theorem 4.1  $f$  cannot be square-free in  $\overline{\mathbb{F}}[t, x_1]$ . This is a contradiction to our assumptions which shows that Eq. 5.9 does not hold. Hence  $\Psi_l(f)$  is separable in  $t$  as was to show.  $\square$

We now prove Theorem 5.2.

*Proof.* Let  $\tilde{f} := \Psi_l(f)$ . We prove this by applying Lemma 3.2 to  $\tilde{f}$ . By Lemma 3.2 there exists a polynomial  $P \in \mathbb{F}[x_1, x_2]$  such that

$$(5.10) \quad \{(\beta_1, \beta_2) \in \mathbb{F}^2 : \tilde{f}(t, \beta_1, \beta_2) \text{ is not separable}\} \subseteq \{(\beta_1, \beta_2) \in \mathbb{F}^2 : P(\beta_1, \beta_2) = 0\}.$$

But

$$(5.11) \quad \tilde{f}(t, \beta_1, \beta_2) = f(t, a(t)\beta_1 + b(t)\beta_2 + c(t)).$$

From Eq. 5.10 and Eq. 5.11 it follows that Eq. 5.2 holds as required.

Let  $C(N)$  be sufficiently large such that  $C(N) \geq \|(a, b, c)\| \deg_x f + \deg_t f$  holds. Assume at least one of (1) and (2) in Theorem 5.2 holds. It follows that at least one of (1) and (2) in Lemma 5.7 holds. Thus by Lemma 5.7  $\tilde{f}$  is separable in  $t$ . Hence by Lemma 3.2  $P$  is non-zero.

Now,

$$\deg_t \tilde{f} \leq \|(a, b, c)\| \deg_x f + \deg_t f$$

and

$$\deg_{\vec{x}} \tilde{f} \leq \deg_x f,$$

where  $\deg_{\vec{x}}$  denotes the total degree of  $\tilde{f}$  as polynomial in variables  $x_1, x_2$ . Assigning this into the bound which is given by Eq. 3.6 in Lemma 3.2 we get

$$\deg P \leq 2(\deg_t \tilde{f} - 1) \deg_{\vec{x}} \tilde{f} \leq$$

$$(2(\|(a, b, c)\| \deg_x f + \deg_t f) - 1) \deg_x f.$$

Let  $\tilde{C}(N)$  be such that  $\tilde{C}(N) \geq (2(\|(a, b, c)\| \deg_x f + \deg_t f) - 1) \deg_x f$ , then  $\deg P \leq \tilde{C}(N)$  holds. Since  $P$  depends on  $f$ , and hence on  $a, b, c$  and  $f$ , we may denote it by  $P_{f,a,b,c}$ .  $\square$

## 6. SQUARE FREE VALUES OF MULTIVARIATE POLYNOMIALS

The main result of this section is a generalization of Theorem 2.1 to multivariate polynomials. The proof of the main result is a direct generalization of the proof of Theorem 2.1. We use this generalization to estimate the number of square-free values of a multivariate polynomial  $f$  at the set  $\mathcal{M}_{m_1} \times \cdots \times \mathcal{M}_{m_d}$  where  $m_1, \dots, m_d \in \mathbb{N}$ ,  $\deg_t f, \deg_{\vec{x}} f$  are fixed and  $q \rightarrow \infty$ . This result is stated in Corollary 2.6 in Section 2.2.

We first fix more notation for this section. Let  $D$  be a unique factorization domain. For each  $i, 1 \leq i \leq d$ , let  $a_i, b_i, c_i \in D$ . Let  $l_i \in D[x_i, x_{d+i}]$  defined by  $l_i = a_i x_i + b_i x_{d+i} + c_i$ . We define a  $D$ -homomorphism  $\Psi_l : D[x_1, \dots, x_d] \rightarrow D[x_1, \dots, x_{2d}]$  by  $x_i \mapsto l_i$  for each  $i, 1 \leq i \leq d$ . Equivalently,  $\Psi_l$  is defined by  $f(x_1, \dots, x_d) \mapsto f(l_1(x_1, x_{d+1}), \dots, l_d(x_d, x_{2d}))$ . Throughout this section  $a_1, b_1, c_1, \dots, a_d, b_d, c_d$  denote elements in  $D$  and  $\Psi_l$  is the homomorphism as defined above. In the first lemma  $D$  is assumed to be any unique factorization domain, while in the rest of this section  $D$  is assumed to be  $\mathbb{F}[t]$ .

To short the notation, we use vector notation. Hence  $\vec{a} = (a_1, \dots, a_d), \vec{b} = (b_1, \dots, b_d), \vec{c} = (c_1, \dots, c_d)$ . Also  $\vec{\beta} = (\beta_1, \dots, \beta_{2d})$  where  $\beta_1, \dots, \beta_{2d} \in \mathbb{F}$ . For  $D = \mathbb{F}[t]$  we define the following notation which generalizes the corresponding notation for a univariate polynomial.

$$(6.1) \quad \mathcal{P}_{\mathbb{F},d}(\vec{a}, \vec{b}, \vec{c}) := \mathcal{P}_{\mathbb{F}}(a_1, b_1, c_1) \times \cdots \times \mathcal{P}_{\mathbb{F}}(a_d, b_d, c_d) = \\ \{(a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, \\ a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t)) \in \mathbb{F}[t]^d : \beta_1, \dots, \beta_{2d} \in \mathbb{F}\}.$$

**6.1. Lemmas needed for the proof of the main result.** We start by generalizing the lemmas of Section 5.3.

**Lemma 6.1** (Generalization of Lemma 5.5 part 4). *Let  $D$  be a unique factorization domain. Let  $b_1, a_1, c_1, \dots, b_d, a_d, c_d \in D$ , such that for each  $i, 1 \leq i \leq d$ ,  $\gcd(a_i, b_i) = 1$ , and let  $\Psi_l : D[x_1, \dots, x_d] \rightarrow D[x_1, \dots, x_{2d}]$  be a  $D$ -homomorphism defined as in the beginning of Section 6. If  $f$  is irreducible in  $D[x_1, \dots, x_d]$  then  $\Psi_l(f)$  is irreducible in  $D[x_1, \dots, x_{2d}]$ .*

*Proof.* The proof follows by applying Lemma 5.5 part 4 inductively. The case where  $d = 1$  is proved by Lemma 5.5 part 4. Suppose the lemma holds for  $d - 1$ . Let  $\tilde{f} \in D[x_1, \dots, x_{2d-1}]$  defined by

$$(6.2) \quad \tilde{f} = f(a_1 x_1 + b_1 x_{d+1} + c_1, a_2 x_2 + b_2 x_{d+2} + c_2, \dots, \\ a_{d-1} x_{d-1} + b_{d-1} x_{2d-1} + c_{d-1}, x_d).$$

We first view  $f$  as a polynomial in the variables  $x_1, \dots, x_{d-1}$  over the domain  $D[x_d]$ . Then by the induction assumption  $\tilde{f}$  is irreducible. Now view  $\tilde{f}$  as a univariate polynomial in the variable  $x_d$ . Then by Lemma 5.5 part 4 it follows that  $\tilde{f}(x_1, \dots, x_{d-1}, a_d x_d + b_d x_{2d} + c_d)$  is irreducible. But  $\Psi_l(f) = \tilde{f}(x_1, \dots, x_{d-1}, a_d x_d + b_d x_{2d} + c_d)$ . Hence the lemma follows.  $\square$

**Lemma 6.2** (Generalization of Lemma 5.6). *Let  $\mathbb{F}$  be a field of positive characteristic  $p$ . Let  $f \in \mathbb{F}(t)(x_1, \dots, x_d)$ . Let  $a_1, b_1, c_1, \dots, a_d, b_d, c_d \in \mathbb{F}(t)$  such that for each  $i$ ,  $1 \leq i \leq d$   $a_i \neq 0$  and  $\frac{b_i}{a_i} \notin \mathbb{F}(t^p)$ . Suppose*

$$(6.3) \quad f(t, a_1(t)x_1 + b_1(t)x_{d+1} + c_1(t), \dots, a_d(t)x_d + b_d(t)x_{2d} + c_d(t)) \in \mathbb{F}(t^p, x_1, \dots, x_{2d}),$$

then  $f \in \mathbb{F}(t^p, x_1^p, \dots, x_d^p)$ .

*Proof.* As in the proof of Lemma 5.6, we first note that it is sufficient to prove the lemma in the case where for each  $i$ ,  $1 \leq i \leq d$   $a_i = 1$  and  $c_i = 0$ . This is because if we define  $\overline{f}(t, x_1, \dots, x_d) := f(t, a_1(t)x_1 + c_1(t), \dots, a_d(t)x_d + c_d(t))$ , and define  $\overline{a_i} := 1, \overline{b_i} := \frac{b_i}{a_i}, \overline{c_i} := 0$ , then  $f(t, a_1(t)x_1 + b_1(t)x_{d+1} + c_1(t), \dots, a_d(t)x_d + b_d(t)x_{2d} + c_d(t)) = \overline{f}(t, x_1 + \frac{b_1}{a_1}(t)x_{d+1}, \dots, x_d + \frac{b_d}{a_d}(t)x_{2d}) = \overline{f}(t, \overline{a_1}(t)x_1 + \overline{b_1}(t)x_{d+1} + \overline{c_1}(t), \dots, \overline{a_d}(t)x_d + \overline{b_d}(t)x_{2d} + \overline{c_d}(t))$ . Proving the lemma for the special case where  $f = \overline{f}$  and for each  $i$ ,  $1 \leq i \leq d$   $a_i = \overline{a_i}, b_i = \overline{b_i}, c_i = \overline{c_i}$  will show that if

$$\overline{f}(t, x_1 + \frac{b_1}{a_1}(t)x_{d+1}, \dots, x_d + \frac{b_d}{a_d}(t)x_{2d}) \in \mathbb{F}(t^p)(x_1^p, \dots, x_d^p)$$

then

$$\overline{f} \in \mathbb{F}(t^p)(x_1^p, \dots, x_d^p),$$

i.e.  $\overline{f} = g(t^p, x_1^p, \dots, x_d^p)$  for some  $g \in \mathbb{F}(t)(x_1, \dots, x_d)$ . But then

$$(6.4) \quad f(t, x_1, \dots, x_d) = \overline{f}\left(t, \frac{x_1 - c_1(t)}{a_1(t)}, \dots, \frac{x_d - c_d(t)}{a_d(t)}\right) = g\left(t^p, \left(\frac{x_1 - c_1(t)}{a_1(t)}\right)^p, \dots, \left(\frac{x_d - c_d(t)}{a_d(t)}\right)^p\right) \in \mathbb{F}(t^p)(x_1^p, \dots, x_d^p).$$

Hence from now on we can assume that for each  $i$ ,  $1 \leq i \leq d$   $a_i = 1$  and  $c_i = 0$ . Now, let

$$\tilde{f} := f(t, x_1 + b_1(t)x_{d+1}, \dots, x_d + b_d(t)x_{2d}).$$

Suppose  $\tilde{f} \in \mathbb{F}(t^p)(x_1, \dots, x_{2d})$ . Then by the chain rule

$$\begin{aligned}
 (6.5) \quad 0 &= \frac{\partial \tilde{f}}{\partial t}(t, x_1, \dots, x_d) = \\
 &\sum_{i=1}^d \frac{\partial f}{\partial x_i}(t, x_1 + b_1(t)x_{d+1}, \dots, x_d + b_d(t)x_{2d})b'_i(t)x_{d+i} + \\
 &\quad \frac{\partial f}{\partial t}(t, x_1 + b_1(t)x_{d+1}, \dots, x_d + b_d(t)x_{2d}).
 \end{aligned}$$

By a change of variables  $x_i = x_i - b_i(t)x_{d+i}, \forall i, 1 \leq i \leq d$  we get:

$$0 = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(t, x_1, \dots, x_d)b'_i(t)x_{d+i} + \frac{\partial f}{\partial t}(t, x_1, \dots, x_d).$$

We view the above as a polynomial in variables

$$x_{d+1}, \dots, x_{2d}$$

over  $\mathbb{F}(t)(x_1, \dots, x_d)$ . By equating the coefficients of this polynomial to 0 we get the following equations

$$(6.6) \quad \frac{\partial f}{\partial t}(t, x_1, \dots, x_d) = 0$$

$$(6.7) \quad \frac{\partial f}{\partial x_i}(t, x_1, \dots, x_d)b'_i(t) = 0, \forall i, 1 \leq i \leq d$$

The lemma follows from Eq. 6.6 and Eq. 6.7 above and by the assumption that  $\frac{b_i}{a_i} = b_i \notin \mathbb{F}(t^p)$ , hence  $b'_i \neq 0$ , for each  $i, 1 \leq i \leq d$ . □

**Lemma 6.3** (Generalization of Lemma 5.7). *Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  be square-free in  $\overline{\mathbb{F}}[t][x_1, \dots, x_d]$ . Let  $a_1, b_1, c_1, \dots, a_d, b_d, c_d \in \mathbb{F}[t]$  such that for each  $i, 1 \leq i \leq d$ ,  $\gcd(a_i, b_i) = 1$ . Let  $\Psi_l : \mathbb{F}[t][x_1, \dots, x_d] \rightarrow \mathbb{F}[t][x_1, \dots, x_{2d}]$  be the  $\mathbb{F}[t]$ -homomorphism as defined in the beginning of Section 6. If at least one of the following holds then  $\Psi_l(f)$  is separable in  $t$ .*

- (1)  $p = 0$  or  $p > \|(b_1, a_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\bar{x}} f + \deg_t f$ .
- (2) For each  $i, 1 \leq i \leq d$   $\frac{b_i}{a_i} \notin \mathbb{F}(t^p)$  where  $a_i \neq 0$ .

*Proof.* Let  $f = \prod_{i=1}^k f_i$  be a factorization of  $f$  into irreducible factors. Then by homomorphism properties

$$(6.8) \quad \Psi_l(f) = \prod_{i=1}^k \Psi_l(f_i).$$

By Lemma 6.1 Eq. 6.8 provides a factorization of  $\Psi_l(f)$  into irreducible factors in  $\mathbb{F}[t][x_1, \dots, x_{2d}]$ . We now show that  $\Psi_l(f)$  is square-free. Suppose  $\Psi_l(f_i) = \Psi_l(f_j)\alpha$  where  $i \neq j$  and  $\alpha \in \mathbb{F}$ . Since  $\Psi_l(\alpha) = \alpha$  we have  $\Psi_l(f_i) = \Psi_l(f_j)\Psi_l(\alpha) = \Psi_l(f_j\alpha)$ . But  $\Psi_l$  is injective, since Lemma 6.1 implies in particular that the kernel of  $\Psi_l$  is trivial. Thus we conclude that

$f_i = f_j \alpha$ . But that is a contradiction to the assumption that  $f$  is square-free. Hence  $\Psi_l(f)$  is square-free. If  $p = 0$  then we are done, since by Corollary 4.3 part 1  $\Psi_l(f)$  being square-free implies that  $\Psi_l(f)$  is separable in  $t$ .

Now suppose  $p > 0$ , and suppose on the contrary that  $\Psi_l(f)$  is not separable as polynomial in  $t$ . Then by Corollary 4.3 part 2  $\Psi_l(f)$  has an irreducible factor in  $\mathbb{F}[t^p][x_1, \dots, x_{2d}]$  where its degree in  $t$  is not zero. Hence we can assume without loss of generality that for some  $j$ ,  $1 \leq j \leq k$

$$(6.9) \quad \Psi_l(f_j) \in \mathbb{F}[t^p][x_1, \dots, x_{2d}], \text{ where } \deg_t \Psi_l(f_j) > 0.$$

We will now show that Eq. 6.9 cannot hold. We split the proof, depending on which of (1) or (2) holds.

Suppose (1) holds. Then Eq. 6.9 cannot hold since for any  $j$ ,  $1 \leq j \leq k$   $\deg_t(\Psi_l(f_j)) \leq \deg_t(\Psi_l(f)) \leq \|(b_1, a_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\bar{x}} f + \deg_t f < p$  where the last inequality is by our assumption on  $p$ .

Suppose (2) holds. Since  $\frac{b_i}{a_i} \notin \mathbb{F}(t^p) \forall i$ ,  $1 \leq i \leq d$ , by Lemma 6.2  $f_j \in \mathbb{F}[t^p, x_1^p, \dots, x_d^p]$ . Now by the equivalent conditions in Theorem 4.1  $f$  cannot be square-free in  $\overline{\mathbb{F}}[t, x_1, \dots, x_d]$ . This is a contradiction to our assumptions which shows that Eq. 6.9 does not hold. Hence  $\Psi_l(f)$  is separable in  $t$ .  $\square$

**6.2. The main theorem for a multivariate polynomial.** We now state the main theorem for a multivariate polynomial over a finite field which generalizes Theorem 2.1.

**Theorem 6.4** (Generalization of Theorem 2.1). *Let  $f \in \mathbb{F}_q[t][x_1, \dots, x_d]$  be a square-free polynomial. Let  $a_1, b_1, c_1, \dots, a_d, b_d, c_d \in \mathbb{F}_q[t]$  such that for each  $i$ ,  $1 \leq i \leq d$   $\gcd(a_i, b_i) = 1$ . Let  $N \in \mathbb{N}$ . Assume*

$$\deg_{\bar{x}} f, \deg_t f, \|(a_1, b_1, c_1, \dots, a_d, b_d, c_d)\| \leq N.$$

*Assume that at least one of the following holds*

- (1)  $p > C(N)$  where  $C(N) \in \mathbb{N}$  is a constant which depends only on  $N$ .
- (2) For each  $i$ ,  $1 \leq i \leq d$   $\frac{b_i}{a_i} \notin \mathbb{F}(t^p)$  where  $a_i \neq 0$ .

*Then while  $N$  remains fixed, we have:*

$$(6.10) \quad \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}))}{\#\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c})} = 1 + O\left(\frac{1}{q}\right), \quad \text{as } q \rightarrow \infty.$$

*In particular, if  $q$  is sufficiently large with respect to  $N$  then there exist  $\vec{\beta} \in \mathbb{F}_q^{2d}$  such that  $f(t, c_1(t) + a_1(t)\beta_1 + b_1(t)\beta_{d+1}, \dots, c_d(t) + a_d(t)\beta_d + b_d(t)\beta_{2d})$  is square-free.*

Let  $\kappa_1, \dots, \kappa_d \in \mathbb{N}$  where for each  $i$ ,  $1 \leq i \leq d$   $\kappa_i \not\equiv 0 \pmod{p}$ . Example 2.7 in Section 2.2 is the specific case of Theorem 6.4 where  $a_i = 1$ , and  $b_i = t^{\kappa_i}$ ,  $\forall i$ ,  $1 \leq i \leq d$ .

As we did in the case of a univariate polynomial, we prove Theorem 6.4 by stating and proving an algebraic theorem which holds for a general field



$\mathbb{F}$ , and showing that Theorem 6.4 follows from the algebraic theorem in the case where  $\mathbb{F}$  is a finite field. The following theorem holds for any field  $\mathbb{F}$ .

**Theorem 6.5** (Generalization of Theorem 5.2). *Let  $\mathbb{F}$  be a field, and let  $\overline{\mathbb{F}}$  be an algebraic closure of  $\mathbb{F}$ . Let  $f \in \mathbb{F}[t][x_1, \dots, x_d]$  be a polynomial which is square-free in  $\overline{\mathbb{F}}[t][x_1, \dots, x_d]$ . Let  $a_1, b_1, c_1, \dots, a_d, b_d, c_d \in \mathbb{F}[t]$  such that for each  $i$ ,  $1 \leq i \leq d$   $\gcd(a_i, b_i) = 1$ . Assume*

$$\deg_{\vec{x}} f, \deg_t f, \|(a_1, b_1, c_1, \dots, a_d, b_d, c_d)\| \leq N.$$

*Then there exists a polynomial  $P_{f, \vec{a}, \vec{b}, \vec{c}} \in \mathbb{F}[x_1, \dots, x_{2d}]$  which depends on  $b_1, a_1, c_1, \dots, b_d, a_d, c_d$  and  $f$  such that*

$$(6.11) \quad \{\vec{\beta} \in \mathbb{F}^{2d} : f(t, a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t)) \text{ is not separable}\} \\ \subseteq \{\vec{\beta} \in \mathbb{F}^{2d} : P_{f, \vec{a}, \vec{b}, \vec{c}}(\beta_1, \dots, \beta_{2d}) = 0\}.$$

*Moreover, there exists a constant  $\tilde{C}(N)$  which depends only on  $N$  such that*

$$\deg P_{f, \vec{a}, \vec{b}, \vec{c}} \leq \tilde{C}(N).$$

*$P_{f, \vec{a}, \vec{b}, \vec{c}}$  is non-zero if at least one of the following holds:*

- (1)  $p = 0$  or  $p > C(N)$  where  $C(N)$  is a constant which depends only on  $N$ .
- (2) For each  $i$ ,  $1 \leq i \leq d$   $\frac{b_i}{a_i} \notin \mathbb{F}(t^p)$  where  $a_i \neq 0$ .

*Proof.* Let  $\tilde{f} \in \mathbb{F}[x_1, \dots, x_{2d}]$ ,  $\tilde{f} := \Psi_l(f)$ . We prove this by applying Lemma 3.2 to  $\tilde{f}$ . By Lemma 3.2 there exists a polynomial  $P \in \mathbb{F}[x_1, \dots, x_{2d}]$  such that

$$(6.12) \quad \{\vec{\beta} \in \mathbb{F}^{2d} : \tilde{f}(t, \beta_1, \dots, \beta_{2d}) \text{ is not separable}\} \\ \subseteq \{\vec{\beta} \in \mathbb{F}^{2d} : P(\beta_1, \dots, \beta_{2d}) = 0\}.$$

But

$$(6.13) \quad \tilde{f}(t, \beta_1, \dots, \beta_{2d}) = f(t, a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, \\ a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t)).$$

From Eq. 6.12 and Eq. 6.13 it follows that Eq. 6.11 holds as required.

Let  $C(N)$  be such that  $C(N) \geq \|(b_1, a_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\vec{x}} f + \deg_t f$ . Suppose at least one of (1) or (2) in Theorem 6.5 holds. It follows that at least one of (1) or (2) in Lemma 6.3 holds. Hence by Lemma 6.3  $\tilde{f}$  is separable in  $t$ . Hence by Lemma 3.2  $P$  is non-zero.

Now,

$$\deg_t \tilde{f} \leq \|(a_1, b_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\vec{x}} f + \deg_t f$$

and

$$\deg_{\vec{x}} \tilde{f} \leq \deg_{\vec{x}} f.$$

Assigning this into the bound which is given by Eq. 3.6 in Lemma 3.2 we get

$$(6.14) \quad \deg P \leq 2(\deg_t \tilde{f} - 1) \deg_{\vec{x}} \tilde{f} \leq (2(\|(a_1, b_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\vec{x}} f + \deg_t f) - 1) \deg_{\vec{x}} f.$$

Let  $\tilde{C}(N)$  be such that  $\tilde{C}(N) \geq (2(\|(a_1, b_1, c_1, \dots, a_d, b_d, c_d)\| \deg_{\vec{x}} f + \deg_t f) - 1) \deg_{\vec{x}} f$ , then  $\deg P \leq \tilde{C}(N)$ . Since  $P$  depends on  $\tilde{f}$ , and hence on  $a_1, b_1, c_1, \dots, a_d, b_d, c_d$  and  $f$ , we may denote it by  $P_{f, \vec{a}, \vec{b}, \vec{c}}$   $\square$

We now show that Theorem 6.4 follows from Theorem 6.5 in the case where  $\mathbb{F}$  is a finite field.

*Proof.* First, over a finite field the requirement in Theorem 6.5 that  $f$  is square-free in  $\overline{\mathbb{F}}[t][x_1, \dots, x_d]$  can be replaced by the requirement that  $f$  is square-free. Over a finite field the two requirements are equivalent by Corollary 4.2, since  $\mathbb{F}_q$  is a perfect field. Likewise, by the same corollary over a finite field  $f(t, a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t))$  being separable and  $f(t, a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t))$  being square-free can be used interchangeably, since the two are equivalent over a perfect field.

With the assumptions and definitions as in Theorem 6.4, we now show why the estimate in Eq. 6.10 of Theorem 6.4 follows. The error  $E(q)$  of estimating

$$\frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}))}{\#\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c})}$$

by 1 is given by

$$(6.15) \quad E(q) = \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f)^c \cap \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}))}{\#\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c})} = \frac{1}{q^{2d}} \# \{ \vec{\beta} \in \mathbb{F}_q^{2d} : f(t, a_1(t)\beta_1 + b_1(t)\beta_{d+1} + c_1(t), \dots, a_d(t)\beta_d + b_d(t)\beta_{2d} + c_d(t)) \text{ is not square-free} \} \leq \frac{\# \{ \vec{\beta} \in \mathbb{F}_q^{2d} : P_{f, \vec{a}, \vec{b}, \vec{c}}(\beta_1, \dots, \beta_{2d}) = 0 \}}{q^{2d}}.$$

Assume first that  $a_1, b_1, c_1, \dots, a_d, b_d, c_d$  are such that  $P_{f, \vec{a}, \vec{b}, \vec{c}}$  is nonzero. Then applying Lemma 5.1

$$E(q) \leq \frac{\tilde{C}(N)q^{2d-1}}{q^{2d}}.$$

Hence keeping  $N$  fixed while  $q \rightarrow \infty$ ,  $E(q) = O\left(\frac{1}{q}\right)$ .

It remains to show why it follows from the assumptions of Theorem 6.4 that  $P_{f, \vec{a}, \vec{b}, \vec{c}}$  is non-zero. The latter is true since we assume at least one of (1) and (2) in Theorem 6.4 holds, but by letting  $C(N)$  in Theorem 6.4 be

the same as  $C(N)$  in Theorem 6.5, (1) and (2) in Theorem 6.4 are the same as (1) and (2) in Theorem 6.5.  $\square$

**6.3. Proof of Corollary 2.6.** We now show that Corollary 2.6 which was stated in Section 2.2 follows from Theorem 6.4. For each choice of  $c_1, \dots, c_d$  where  $c_1, \dots, c_d$  are monic with the first two coefficients zero, we use Theorem 6.4 to estimate the number of square-free values of  $f$  which are obtained by perturbing the first two coefficients of each of  $c_1, \dots, c_d$ . By summing over all possible choices for  $c_1, \dots, c_d$ , we get the result which is stated in Corollary 2.6. We now show this in more details.

*Proof.* For each  $i$ ,  $1 \leq i \leq d$  let  $C_i$  be the set of monic polynomials of degree  $m_i$  with the first two coefficients zero. Explicitly,

$$C_i = \left\{ c \in \mathbb{F}_q[t] : c(t) = t^{m_i} + \sum_{j=2}^{m_i-1} a_j t^j, a_2, \dots, a_{m_i-1} \in \mathbb{F}_q \right\}.$$

Let  $c_1 \in C_1, \dots, c_d \in C_d$ , and for each  $i$ ,  $1 \leq i \leq d$  let  $a_i = t, b_i = 1$ . Then

$$(6.16) \quad \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}) = \{(t\beta_1 + \beta_{d+1} + c_1(t), \dots, t\beta_d + \beta_{2d} + c_d(t)) \in \mathbb{F}[t]^d : \beta_1, \dots, \beta_{2d} \in \mathbb{F}_q\}.$$

For each choice of  $\vec{c} \in \mathbb{F}[t]^d$  where  $c_1 \in C_1, \dots, c_d \in C_d$ ,  $\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c})$  is the set of polynomials obtained by perturbing the first two coefficients of  $c_1, \dots, c_d$ . Each element of  $\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d}$  is an element of  $\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c})$  for exactly one choice of  $\vec{c}$ . Stating this differently

$$\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d} = \bigcup_{c_1 \in C_1, \dots, c_d \in C_d} \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}).$$

where the union is disjoint. Hence

$$\#((\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d}) \cap \mathcal{S}_{\mathbb{F}_q, d}(f)) = \sum_{c_1 \in C_1, \dots, c_d \in C_d} \#(\mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}) \cap \mathcal{S}_{\mathbb{F}_q, d}(f)).$$

The size of each  $C_i$  is  $\#C_i = q^{m_i-2}$ . Hence the number of elements in the last sum is  $q^{m-2d}$  where  $m = m_1 + \dots + m_d$ . Thus

$$(6.17) \quad \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap (\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d}))}{\#\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d}} = \sum_{c_1 \in C_1, \dots, c_d \in C_d} \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}))}{\#\mathcal{M}_{m_1} \times \dots \times \mathcal{M}_{m_d}} = \frac{1}{q^{m-2d}} \sum_{c_1 \in C_1, \dots, c_d \in C_d} \frac{\#(\mathcal{S}_{\mathbb{F}_q, d}(f) \cap \mathcal{P}_{\mathbb{F}_q, d}(\vec{a}, \vec{b}, \vec{c}))}{q^{2d}} = \frac{1}{q^{m-2d}} q^{m-2d} \left(1 + O\left(\frac{1}{q}\right)\right),$$

where the last equality follows by Theorem 6.4  $\square$

## APPENDIX A. PROOF OF THEOREM 4.1 AND ITS COROLLARIES

**Lemma A.1.** *Let  $\mathbb{F}$  be a field, and  $L \supseteq \mathbb{F}$  an algebraic field extension of  $\mathbb{F}$ . Let  $h \in L[x_1, \dots, x_d]$  be nonconstant and irreducible in  $L[x_1, \dots, x_d]$ . Then there exists  $h_m \in \mathbb{F}[x_1, \dots, x_d]$  such that:*

- (1)  $h|h_m$ .
- (2) If  $f \in \mathbb{F}[x_1, \dots, x_d]$  and  $h|f$  then  $h_m$  divides  $f$  in  $\mathbb{F}[x_1, \dots, x_d]$ .

**Remark 3.** For any  $i$ ,  $1 \leq i \leq d$ ,  $\deg_{x_i} h_m > 0$  implies  $\deg_{x_i} h > 0$ . To show this, let  $h_m$  which existence is provided by Lemma A.1 applied to  $h$  as an element in  $L[x_1, \dots, x_d]$ . By applying Lemma A.1 to  $h$  as an element in  $L[x_{i_1}, \dots, x_{i_k}]$  where  $x_{i_1}, \dots, x_{i_k}$  are the variables that appear in  $h$  we conclude that there exists  $\hat{h}_m$  in  $\mathbb{F}[x_{i_1}, \dots, x_{i_k}]$  which is divided by  $h_m$ . Hence the variables that appear in  $h_m$  are in the set  $\{x_{i_1}, \dots, x_{i_k}\}$ .

*Proof.* In other words the lemma asserts that  $\langle h \rangle \cap \mathbb{F}[x_1, \dots, x_d]$  is a nonempty principal ideal in  $\mathbb{F}[x_1, \dots, x_d]$ . We prove first the case where  $d = 1$ . Let  $\alpha \in \bar{\mathbb{F}}$  be a root of  $h$ . Since  $\alpha$  is algebraic over  $\mathbb{F}$  there exists  $g \in \mathbb{F}[x_1]$  such that  $g(\alpha) = 0$ . Since  $L[x_1]$  is a principal ideal domain, there exists  $r \in L[x_1]$  such that  $\langle r \rangle = \langle h, g \rangle$ . Also,  $1 \notin \langle r \rangle$  since  $\alpha$  is a root of every polynomial in  $\langle r \rangle$ . Hence  $r$  is not invertible. But  $h$  is irreducible, hence  $h|r$ , and  $\langle h \rangle = \langle r \rangle = \langle h, g \rangle$ . Thus  $g \in \langle h \rangle \cap \mathbb{F}[x_1]$  and so  $\langle h \rangle \cap \mathbb{F}[x_1]$  is not empty. In addition,  $\langle h \rangle \cap \mathbb{F}[x_1]$  is a principal ideal since  $\mathbb{F}[x_1]$  is a principal ideal domain. Hence the lemma follows for the case  $d = 1$ .

Now consider the case where  $d > 1$ . Since  $h$  is nonconstant, we can assume without loss of generality that  $\deg_{x_1} h > 0$ . Denote

$$\tilde{\mathbb{F}} = \mathbb{F}(x_2, \dots, x_d), \tilde{L} = L(x_2, \dots, x_d), D = \mathbb{F}[x_2, \dots, x_d], D_L := L[x_2, \dots, x_d].$$

Then  $\tilde{\mathbb{F}}$  and  $\tilde{L}$  are fields and  $\tilde{L}$  is an algebraic extension of  $\tilde{\mathbb{F}}$ .  $h$  is irreducible also in  $\tilde{L}[x_1]$  by Gauss's lemma for polynomials. It follows from the case  $d = 1$  that there exists  $\tilde{h}_m \in \tilde{\mathbb{F}}[x_1]$  such that

- $h|\tilde{h}_m$ .
- For any  $f \in \mathbb{F}[x_1, \dots, x_d] \subseteq \tilde{\mathbb{F}}[x_1]$  such that  $h|f$ ,  $f = \tilde{h}_m \tilde{u}$  for some  $\tilde{u} \in \tilde{\mathbb{F}}[x_1]$ .

We first show part 2 of the lemma.  $\tilde{\mathbb{F}}$  is the field of fractions of  $D$ . Hence we can write  $\tilde{h}_m = c_{h_m} h_m$  and  $\tilde{u} = c_u u$ , where  $c_{h_m}, c_u \in \tilde{\mathbb{F}}$  and  $h_m, u$  are primitive polynomials in  $D[x_1]$ . Then

$$(A.1) \quad f = \tilde{h}_m \tilde{u} = c_{h_m} h_m c_u u = c_{h_m} c_u h_m u.$$

By Gauss's lemma for polynomials a multiplication of primitive polynomials is a primitive polynomial. Hence  $h_m u$  is a primitive polynomial. Hence by Eq. A.1  $c_{h_m} c_u \in D = \mathbb{F}[x_2, \dots, x_d]$  or otherwise  $f$  would not be a polynomial in  $D[x_1]$  but a rational function. Hence by Eq. A.1  $h_m$  divides  $f$  in  $\mathbb{F}[x_1, \dots, x_d]$ .

To show part 1 of the lemma, there exists  $\tilde{v} \in \tilde{L}[x_1]$  such that  $h\tilde{v} = \tilde{h}_m$ . We can write  $\tilde{v} = c_v v$  where  $c_v \in \tilde{L}$  and  $v \in D_L[x_1]$  is primitive. Hence  $h\tilde{v} = hvc_v = h_m c_{h_m} = \tilde{h}_m$ . Hence

$$(A.2) \quad hv \frac{c_v}{c_{h_m}} = h_m.$$

But  $h$  is irreducible and nonconstant in  $D_L[x_1]$  and in particular primitive, and  $v$  is primitive. Hence  $hv$  is primitive by Gauss's lemma for polynomials. Hence  $\frac{c_v}{c_{h_m}} \in D_L$  or otherwise by Eq. A.2  $h_m$  would not be in  $D[x_1]$ . Hence by Eq. A.2  $h$  divides  $h_m$  in  $L[x_1, \dots, x_d]$ .  $\square$

**Lemma A.2.** *Let  $\mathbb{F}$  be a field, and let  $L \supseteq \mathbb{F}$  be an algebraic field extension of  $\mathbb{F}$ . Let  $f \in \mathbb{F}[x_1, \dots, x_d]$  be a polynomial which is square-free in  $\mathbb{F}[x_1, \dots, x_d]$ . Let  $h \in L[x_1, \dots, x_d]$  be an irreducible polynomial. Suppose  $h^2 | f$ . Let a factorization of  $f$  be  $f = \prod_{i=1}^k f_i$ , where  $f_1, \dots, f_k \in \mathbb{F}[x_1, \dots, x_d]$  are irreducible as elements in  $\mathbb{F}[x_1, \dots, x_d]$ . Then  $h^2 | f_j$  for some  $j$ ,  $1 \leq j \leq k$ . Also, any variable that appears in  $f_j$  appears in  $h$ .*

*Proof.*  $L[x_1, \dots, x_d]$  is a unique factorization domain. Since  $h$  is a prime element in  $L[x_1, \dots, x_d]$  and  $h | f$ , it follows that  $h | f_j$  for some  $1 \leq j \leq k$ . Suppose on the contrary that  $h \nmid f_j$ . By Lemma A.1 there exists  $h_m \in \mathbb{F}[x_1, \dots, x_d]$  such that  $h_m | f_j$  and  $h_m | \frac{f}{f_j}$ . But the last conclusion is a contradiction to the assumption that  $f$  is square-free in  $\mathbb{F}[x_1, \dots, x_d]$ . Hence  $h^2$  and  $\frac{f}{f_j}$  are co-prime, and  $h^2 | f_j$ . To see that every variable that appears in  $f_j$  appears in  $h$ , as stated in Remark 3 every variable that appears in  $h_m$  appears in  $h$ . Since  $f_j$  is irreducible, hence  $f_j | h_m$ , every variable that appears in  $f_j$  appears in  $h_m$ .  $\square$

We give the following lemma without a proof, only for reference.

**Lemma A.3.** *Let  $\mathbb{F}$  be a field. Let  $f \in \mathbb{F}[x]$  be an irreducible polynomial.*

- (1) *If  $\text{Char}(\mathbb{F}) = 0$ , then  $f$  is separable.*
- (2) *If  $\mathbb{F}$  is of positive characteristic  $p$ , and  $f$  is non-separable then  $f \in \mathbb{F}[x^p]$ .*

For a proof of part 1 of the lemma above see Corollary 34 in chapter 13 of [2]. As to part 2, in fact, a stronger statement holds which is that there exists a unique  $k \geq 0$  such that  $f = f_{\text{sep}}(x^{p^k})$  where  $f_{\text{sep}} \in \mathbb{F}[x]$  is a separable polynomial. For a proof see proposition 38 in chapter 13 of [2]. For our usage the weaker statement in Lemma A.3 appears to be sufficient.

**Lemma A.4.** *Let  $\mathbb{F}$  be a field of characteristic 0. Let  $f \in \mathbb{F}[x_1, \dots, x_d]$  be a polynomial. Then  $f$  is square-free in  $\overline{\mathbb{F}}[x_1, \dots, x_d]$  if and only if  $f$  is square-free in  $\mathbb{F}[x_1, \dots, x_d]$ .*

*Proof.* On one direction, suppose  $f$  has a nonconstant factor  $g \in \mathbb{F}[x_1, \dots, x_d]$  such that  $g^2 | f$ . Then that holds also when  $g$  and  $f$  are considered as elements of the larger domain  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ .

To see the opposite direction, suppose  $f$  is square-free in  $\mathbb{F}[x_1, \dots, x_d]$  and suppose on the contrary that there exists a non constant irreducible  $h \in \overline{\mathbb{F}}[x_1, \dots, x_d]$  such that  $h^2$  divides  $f$  in  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ . Let  $f = \prod_{i=1}^k f_i$  be a factorization of  $f$  where  $f_i \in \mathbb{F}[x_1, \dots, x_d]$  are irreducible polynomials in  $\mathbb{F}[x_1, \dots, x_d]$ . Suppose  $l \in \mathbb{N}$ ,  $1 \leq l \leq d$  is such that  $\deg_{x_l} h > 0$ . By Lemma A.1 there exists  $j$ ,  $1 \leq j \leq d$  such that  $h^2$  divides  $f_j$ . But that means  $f_j$  is not separable as polynomial in  $x_l$ . We now show this in more details.

Denote

$$K_1 = \mathbb{F}(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_d), \quad K_2 = \overline{\mathbb{F}}(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_d),$$

and denote by  $\overline{K_2}$  the algebraic closure of  $K_2$ , which is also an algebraic closure of  $K_1$ . View  $h$  as an element of  $K_2[x_l]$  and view  $f_j$  as an element of  $K_1[x_l]$ . Suppose  $h = c \prod_{i=1}^I (x_l - \alpha_i)$  where  $c \in K_2$  and  $\alpha_i \in \overline{K_2}$ ,  $1 \leq i \leq I$  is the factorization of  $h$  into linear factors in  $\overline{K_2}[x_l]$ . Then in particular  $(x_l - \alpha_1)^2$  divides  $h^2$  and hence it divides  $f_j$ . Hence  $f_j$  as polynomial in  $K_1[x_l]$  is not separable.

But since  $\text{Char}(K_1) = 0$ , the last cannot hold by Lemma A.3 part 1. A contradiction which shows that such  $h$  does not exist.  $\square$

We now prove Theorem 4.1 and its two corollaries which were stated in Section 4.

*Proof of Theorem 4.1.* (1) $\Rightarrow$ (2): Suppose  $f$  is not square-free as an element in  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ . Then there exists a non constant  $h \in \mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$  such that  $h^2 | f$ . Since  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d] \subseteq \overline{\mathbb{F}}[x_1, \dots, x_d]$ ,  $h^2$  is a square factor of  $f$  also as an element of  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ . (2) $\Rightarrow$ (3): Assume  $f$  is square-free as an element in  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ .  $\mathbb{F}[x_1, \dots, x_d] \subseteq \mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ . Hence by the same argument as in the previous part, it is immediate that  $f$  is square-free as an element in  $\mathbb{F}[x_1, \dots, x_d]$ . It remains to show that  $f$  does not have an irreducible factor  $g \in \mathbb{F}[x_1^p, \dots, x_d^p]$ . Suppose on the contrary there is such factor  $g$ . Then  $g(x_1, \dots, x_d) = h(x_1^p, \dots, x_d^p)$  for some polynomial  $h \in \mathbb{F}[x_1, \dots, x_d]$

$$h(x_1, \dots, x_d) = \sum_{(e_1, \dots, e_d) \in \{0, \dots, n\}^d} c_{e_1, \dots, e_d} \prod_{j=1}^d x_j^{e_j}.$$

Applying the Frobenius automorphism properties we get:

$$(A.3) \quad g(x_1, \dots, x_d) = \sum_{(e_1, \dots, e_d) \in \{0, \dots, d\}^n} c_{e_1, \dots, e_d} \prod_{j=1}^d x_j^{pe_j} = \left( \sum_{(e_1, \dots, e_d) \in \{0, \dots, n\}^d} c_{e_1, \dots, e_d}^{\frac{1}{p}} \prod_{j=1}^d x_j^{e_j} \right)^p$$

Since  $c_{e_1, \dots, e_d}^{\frac{1}{p}} \in \mathbb{F}^{\frac{1}{p}}$  for any  $e_1, \dots, e_d$

$$\sum_{(e_1, \dots, e_d) \in \{0, \dots, n\}^d} c_{e_1, \dots, e_d}^{\frac{1}{p}} \prod_{j=1}^d x_j^{e_j}$$

is a repeated factor of  $f$  in  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ , contradicting the assumption that  $f$  is square-free as an element in  $\mathbb{F}^{\frac{1}{p}}[x_1, \dots, x_d]$ . (3) $\Rightarrow$ (1): Suppose  $f$  is square-free as element in  $\mathbb{F}[x_1, \dots, x_d]$  but is not square-free as an element in  $\overline{\mathbb{F}}[x_1, \dots, x_d]$ . Then there exists an irreducible nonconstant  $h \in \overline{\mathbb{F}}[x_1, \dots, x_d]$  such that  $h^2 | f$ . Let  $f = \prod_{i=1}^k f_i$  be a factorization of  $f$  where  $f_i \in \mathbb{F}[x_1, \dots, x_d]$  are irreducible polynomials in  $\mathbb{F}[x_1, \dots, x_d]$ . By Lemma A.2  $h^2$  divides  $f_j$  for some  $j$ ,  $1 \leq j \leq k$ .

Let  $l \in \mathbb{N}$ ,  $1 \leq l \leq d$ . Let  $K_1 = \mathbb{F}(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_d)$ . Suppose  $\deg_{x_l} f_j > 0$ . Then as stated in Lemma A.2  $\deg_{x_l} h > 0$ . Since  $h^2 | f_j$ , it follows that  $f_j$  is not separable as polynomial in  $K_1[x_l]$ , as we showed in more details at the end of the proof of Lemma A.4. It follows by Lemma A.3 that  $f_j \in K_1[x_l^p]$ . If  $\deg_{x_l} f_j = 0$ , then  $f_j \in K_1[x_l^p]$  holds as well. Hence in any case  $f_j \in K_1[x_l^p]$ . But that is true for any  $l$ ,  $1 \leq l \leq d$ . Hence  $f_j \in \mathbb{F}[x_1^p, \dots, x_d^p]$ .  $\square$

*Proof of Corollary 4.2.* If  $\text{Char}(\mathbb{F}) = 0$ , then this is stated in Lemma A.4. If  $\text{Char}(\mathbb{F}) > 0$ , then since  $\mathbb{F}$  is perfect  $\mathbb{F}^{\frac{1}{p}} = \mathbb{F}$ , where  $\mathbb{F}^{\frac{1}{p}}$  is the field as defined in Eq. 4.1. Hence condition 2 of Theorem 4.1 is equivalent to  $f$  being square-free in  $\mathbb{F}[x_1, \dots, x_d]$ . The corollary follows by the equivalence of conditions 1 and 2 of Theorem 4.1.  $\square$

*Proof of Corollary 4.3.* First, if  $f$  had a square factor in  $\mathbb{F}(x_1, \dots, x_d)[t]$  then by Gauss's lemma for polynomials it would also have a square factor in  $\mathbb{F}[x_1, \dots, x_d][t]$ . Hence we can assume  $f$  is square-free in  $\mathbb{F}(x_1, \dots, x_d)[t]$ .

(1): View  $f$  as a univariate polynomial in  $t$  over  $\mathbb{F}(x_1, \dots, x_d)$ . Since  $\text{Char}(\mathbb{F}(x_1, \dots, x_d)) = 0$  in particular  $\mathbb{F}(x_1, \dots, x_d)$  is perfect. Hence by Corollary 4.2  $f$  being square-free in  $\mathbb{F}(x_1, \dots, x_d)[t]$  implies that  $f$  is square-free in  $\overline{\mathbb{F}(x_1, \dots, x_d)}[t]$  where  $\overline{\mathbb{F}(x_1, \dots, x_d)}$  denotes the algebraic closure of  $\mathbb{F}(x_1, \dots, x_d)$ . Equivalently,  $f$  is separable in  $t$ .

(2): View  $f$  as a univariate polynomial in  $t$  over  $\mathbb{F}(x_1, \dots, x_d)$ . Then  $f$  is square-free as an element in  $\mathbb{F}(x_1, \dots, x_d)[t]$  but not as an element in  $\overline{\mathbb{F}(x_1, \dots, x_d)}[t]$ . Hence by Theorem 4.1  $f$  has an irreducible factor in  $\mathbb{F}(x_1, \dots, x_d)[t^p]$ . In particular the latter is not invertible, hence its degree

in  $t$  is not zero. By multiplying this factor by an element in  $\mathbb{F}(x_1, \dots, x_d)$  we obtain a factor of  $f$  in  $\mathbb{F}[x_1, \dots, x_d][t^p]$ . Let  $g$  be this factor.  $\square$

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